STUDY OF THE BEHAVIOR OF TEMLYAKOV-TYPE INTEGRALS BY THE METHOD OF LINEAR HOMOGENEOUS DIFFERENTIAL OPERATORS OF THE FIRST ORDER

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Abstract

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STUDY OF THE BEHAVIOR OF TEMLYAKOV-TYPE INTEGRALS BY THE METHOD OF LINEAR HOMOGENEOUS DIFFERENTIAL OPERATORS OF THE FIRST ORDER

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Here we give a method that has proved sufficiently effective for studying the behavior of Temlyakov-type integrals in the domain of nonanalyticity. Its essence consists in the application of the properties of differential operators $\widetilde{D}$ of the form $a\,\partial/\partial w+b\,\partial/\partial z+c\,\partial/\partial\bar{w}+d\,\partial/\partial\bar{z}$ and in the generalization to the operators $\widetilde{D}$ of the Newton–Leibniz formula:

$$\widetilde{D}\int_p^g f\,d\xi=\int_p^g \widetilde{D}(f)\,d\xi+\widetilde{D}(g)\,f_{\xi=g}-\widetilde{D}(p)\,f_{\xi=p}.$$

Here a, b, c, d, p, g are arbitrary functions of the variables $w, z, \bar{w}, \bar{z}$, and $f=f(w, z, \bar{w}, \bar{z}, \xi)$.

The study by this method of Temlyakov-type integrals of the first and second kind (see (1))

$$F=\frac{1}{4\pi^2 i}\int_0^1 d\tau\int_0^{2\pi} dt\int_{|\eta|=1}\frac{f(\eta,\tau,t)}{\eta-u}\,d\eta,$$

$$\Phi=\frac{1}{4\pi^2 i}\int_0^1 d\tau\int_0^{2\pi} dt\int_{|\eta|=1}\frac{\eta f(\eta,\tau,t)}{(\eta-u)^2}\,d\eta,$$

where

$$u=\frac{\tau}{r_1(\tau)}\,w+\frac{1-\tau}{r_2(\tau)}\,ze^{it},$$

and of other integrals with an analogous kernel begins with obtaining formulas for passing from multiple integration to repeated integration. Initially these formulas for Temlyakov-type integrals were obtained by Aizenberg (see (2)), and then developed in the work of V. I. Boganov (3). The formula most convenient for the method under consideration is

$$F = \frac{1}{2\pi}\int_0^{\tau_0} d\tau \int_{\alpha_0+\beta}^{2\pi-\alpha_0+\beta} f^+\,dt + \frac{1}{2\pi}\int_{\tau_0}^{\tau_1} d\tau \int_{\alpha+\beta}^{2\pi-\alpha+\beta} f^+\,dt + \frac{1}{2\pi}\int_{\tau_1}^{1} d\tau \int_{\alpha_1+\beta}^{2\pi-\alpha_1+\beta} f^+\,dt$$
$$+\frac{1}{2\pi}\int_0^{\tau_0} d\tau \int_{-\alpha_0+\beta}^{\alpha_0+\beta} f^-\,dt + \frac{1}{2\pi}\int_{\tau_0}^{\tau_1} d\tau \int_{-\alpha+\beta}^{\alpha+\beta} f^-\,dt + \frac{1}{2\pi}\int_{\tau_1}^{1} d\tau \int_{-\alpha_1+\beta}^{\alpha_1+\beta} f^-\,dt,$$

in which f^+ and f^- are the inner integral respectively for $|u| < 1$ and $|u| > 1$, $0 \le \alpha \le \pi$ and $\beta = \arg w - \arg z$ are determined from the equalities $|u|_{t=\beta+\alpha} = |u|_{t=\beta-\alpha} = 1$; τ_0 and τ_1 are the endpoints of the interval $T_3 \subset [0, 1]$, determined by the condition: if $\tau \in T_3$, then the expression $|u| - 1$ changes sign as t varies from 0 to 2π. The values α_0 and α_1 are obtained from α by substituting τ_0 and τ_1, respectively, in place of τ. This formula unifies nine different formulas of Boganov into one and can be verified on the basis of his results.

The second stage of the investigation is the derivation of formulas for the action of an arbitrary operator $\tilde{D}$ on the integrals under study. It is based on a generalization of the Newton–Leibniz rule and on formulas for passing from multiple integration to repeated integration. For an integral of Temlyakov type of the first kind, the following holds.

Theorem 1. *If the density of a Temlyakov integral of the first kind is such that Sokhotskiĭ s formulas are applicable to the inner integral, then*

$$\tilde{D}(F) = \frac{1}{4\pi^2 i}\int_0^1 d\tau \int_0^{2\pi} dt \int_{|\eta|=1} f\tilde{D}\left(\frac{1}{\eta-u}\right) d\eta +$$
$$+\frac{1}{2\pi}\int_{\tau_0}^{\tau_1} \tilde{D}(\beta-\alpha) f_{\substack{\eta=u\\ t=\beta-\alpha}}\, d\tau - \frac{1}{2\pi}\int_{\tau_0}^{\tau_1} \tilde{D}(\beta+\alpha) f_{\substack{\eta=u\\ t=\beta+\alpha}}\, d\tau.$$

An analogous formula also holds for a Temlyakov integral of the second kind Φ.

After these formulas have been obtained, the third stage of the investigation begins. It consists in finding differential operators which play a special role for the integrals under consideration, and in studying the properties of the integrals with the aid of the operators found.

For integrals of Temlyakov type, a special role is played, first of all, by the operator

$$D = w\,\partial/\partial w + z\,\partial/\partial z - \bar{w}\,\partial/\partial \bar{w} - \bar{z}\,\partial/\partial \bar{z},$$

which is defined by the conditions

$$D(|w|) = D(|z|) = D(\beta) = 0.$$

Theorem 2. *Throughout the whole domain of existence of Temlyakov integrals, the action on them by the operator D is equivalent to its action on the kernel.*

With the aid of the operator D, it is possible to establish the relation between Temlyakov integrals of the first and second kinds

$$(1 + D)(F) \equiv L^*(F) = \Phi,$$

which holds throughout the whole domain of their existence and, in the domain of analyticity, turns into the known formula

$$L(F) = F + w\,\partial F/\partial w + z\,\partial F/\partial z = \Phi.$$

For domains of type A, the operator D is not the only operator that can be brought under the sign of Temlyakov integrals (for such domains, α does not depend on τ). The same property for the hypercone $a|w| + b|z| < 1$ is possessed by the operator determined by the conditions

$$D_1\left(\frac{1 - a^2|w|^2 - b^2|z|^2}{|w|\,|z|}\right) = D_1(\arg w) = D_1(\arg z) = 0$$

namely the operator

$$D_1 = (k-1)(w\,\partial/\partial w + \bar{w}\,\partial/\partial \bar{w}) + (k+1)(z\,\partial/\partial z + \bar{z}\,\partial/\partial \bar{z}),$$

in which $k = a^2|w|^2 - b^2|z|^2$, and any linear combinations $pD + gD_1$. Among them, the generalized derivatives deserve the greatest interest:

$$D_w = \frac{\partial}{\partial w} - \frac{1}{w}\left[k\bar{w}\,\frac{\partial}{\partial \bar{w}} + (k+1)\bar{z}\,\frac{\partial}{\partial \bar{z}}\right],$$

$$D_z = \frac{\partial}{\partial z} + \frac{1}{z}\left[(k+1)\bar{w}\,\frac{\partial}{\partial \bar{w}} + k\bar{z}\,\frac{\partial}{\partial \bar{z}}\right],$$

whose action on Temlyakov integrals is equivalent to differentiation of the kernel with respect to the corresponding variable.

Theorem 3. *If the density $f = f(\eta, \tau, t)$ of Temlyakov integrals is defined for all $0 \le \tau \le 1$ and $0 \le t \le 2\pi$ and is analytic in η in the disk $|\eta| < 1 + \varepsilon$ (ε is an arbitrary positive number), then in the domain of non-analyticity these integrals can be represented by uniformly convergent generalized power series*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} \tilde{c}_{mn} w^m z^{n-m},$$

which can be "differentiated" by generalized derivatives arbitrarily many times.

In the resulting generalized power series, the coefficients $\tilde{c}_{mn} = c_{mn}(\alpha, \beta)$ are "generalized constants," i.e., functions satisfying the conditions $D_w(\tilde{c}_{mn}) = D_z(\tilde{c}_{mn}) = 0$. In the domain of analyticity, generalized derivatives turn into ordinary derivatives, generalized constants into ordinary constants, and generalized power series into ordinary power series.

Further results obtained for domains of type A are connected with the operators

$$K_1 = be^{i\alpha}\partial/\partial w - ae^{-i\beta}\partial/\partial z, \qquad K_2 = be^{-i\alpha}\partial/\partial w - ae^{-i\beta}\partial/\partial z,$$

$$\overline{K}_1 = be^{-i\alpha}\partial/\partial \overline{w} - ae^{i\beta}\partial/\partial \overline{z}, \qquad \overline{K}_2 = be^{i\alpha}\partial/\partial \overline{w} - ae^{i\beta}\partial/\partial \overline{z}.$$

With the aid of these differential operators one can obtain three independent second-order differential equations which integrals of Temlyakov type satisfy throughout their domain of existence (generalized Cauchy–Riemann conditions). In terms of the operators K_1, K_2, $\overline{K}_1$, and $\overline{K}_2$, the differential equations are expressed as follows:

$$K_2[\,|w|\,|z|\,\overline{K}_1(F)] = 0, \qquad \overline{K}_2[\,|w|\,|z|\,K_1(F)] = 0,$$

$$K_1[\,|w|\,|z|\,\overline{K}_2(F)] = 0.$$

After reducing them to explicit form for the hypercone $|w| + |z| < 1$, we obtain the system:

$$F''_{w\overline{w}} = F''_{z\overline{z}},$$

$$\frac{w\overline{w} + z\overline{z} - 1}{2}\,(F''_{w\overline{w}} + F''_{z\overline{z}}) + w\overline{z}F''_{wz} + \overline{w}zF''_{\overline{w}\overline{z}} + \overline{w}F'_w + \overline{z}F'_z = 0,$$

$$\overline{w}zF''_{\overline{w}^2} + (w\overline{w} + z\overline{z} - 1)F''_{w\overline{z}} + wzF''_{z^2} + zF'_w + wF'_z = 0.$$

Investigating integrals of Temlyakov type taken over the boundary of the hypersphere $|w|^2 + |z|^2 < 1$, we find that the operators

$$P_0 = w\frac{\partial}{\partial w} + z\frac{\partial}{\partial z}, \qquad P_1 = \frac{1}{w}\frac{\partial}{\partial w} - \frac{1}{z}\frac{\partial}{\partial z},$$

$$\overline{P}_0 = \overline{w}\frac{\partial}{\partial \overline{w}} + \overline{z}\frac{\partial}{\partial \overline{z}}, \qquad \overline{P}_1 = \frac{1}{w}\frac{\partial}{\partial \overline{w}} - \frac{1}{z}\frac{\partial}{\partial \overline{z}}.$$

play a special role.

When the operators $\overline{P}_0$ and $\overline{P}_1$ act on Temlyakov integrals of the first kind, constructed for functions $f(w, z, \overline{w}, \overline{z})$, i.e., having density

$$f\left(\sqrt{\tau\eta}, \sqrt{1-\tau\eta}\, e^{-it}, \sqrt{\tau\eta^{-1}}, \sqrt{1-\tau\eta^{-1}}\, e^{it}\right),$$

we obtain the formulas

$$\overline{P}_0(F) = -\frac{\varepsilon}{2\pi i}\int_{|\zeta|=1} f(w_\zeta, z_\zeta, \overline{w}_\zeta, \overline{z}_\zeta)\,\frac{d\zeta}{\zeta},$$

$$\overline{P}_1(F) = -\frac{\sqrt{\varepsilon(1-\varepsilon)}}{|w|\,|z|}\frac{1}{2\pi i}\int_{|\zeta|=1} f(w_\zeta, z_\zeta, \overline{w}_\zeta, \overline{z}_\zeta)\,d\zeta,$$

where

$$\varepsilon = \frac{1}{|w|^2+|z|^2}, \qquad w_\zeta = w\varepsilon + \frac{w|z|}{|w|}\sqrt{\varepsilon(1-\varepsilon)}\,\zeta, \qquad z_\zeta = z\varepsilon - \frac{z||w|}{|z|}\sqrt{\varepsilon(1-\varepsilon)}\,\zeta,$$

from which it follows:

1) Temlyakov integrals outside the hypersphere satisfy two first-order differential equations: $\overline{P}_1(F) = 0$ and $L(F) = 0$;
2) if $f(w, z, \overline{w}, \overline{z})$ is a function continuously differentiable on the boundary of the hypersphere, then the integral constructed for it satisfies the second-order differential equation

$$(w\overline{w}-1)F''_{w\overline{w}} + (z\overline{z}-1)F''_{z\overline{z}} + w\overline{z}F''_{w\overline{z}} + \overline{w}zF''_{\overline{w}z} + \overline{w}F'_w + \overline{z}F'_z = 0;$$

3) Temlyakov-type integrals can be analytic functions even outside the hypersphere.

For example, forming a Temlyakov-type integral of the first kind for the function $w/\bar{w}$, or, what is the same, taking the density $f = \eta^2$, we obtain

$$\bar{P}_0(F) = -\frac{\varepsilon}{2\pi i}\int_{|\zeta|=1}\frac{w\varepsilon + \dfrac{w|z|}{|w|}\sqrt{\varepsilon(1-\varepsilon)}\,\zeta}{\zeta\left(\bar{w}\varepsilon + \dfrac{w|z|}{|w|}\sqrt{\varepsilon(1-\varepsilon)}\,\dfrac{1}{\zeta}\right)}\,d\zeta,$$

$$\bar{P}_1(F) = -\frac{\sqrt{\varepsilon(1-\varepsilon)}}{|w|\,|z|}\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{w\varepsilon + \dfrac{w|z|}{|w|}\sqrt{\varepsilon(1-\varepsilon)}\,\zeta}{\bar{w}\varepsilon + \dfrac{w|z|}{|w|}\sqrt{\varepsilon(1-\varepsilon)}\,\dfrac{1}{\zeta}}\,d\zeta.$$

From this it is easy to establish that in the domain $|z| > 1$ the conditions $\bar{P}_0(F) = \bar{P}_1(F) = 0$ are satisfied, or, what is the same, the Cauchy–Riemann conditions

$$\partial F/\partial \bar{w} = \partial F/\partial \bar{z} = 0.$$

The results obtained show that Temlyakov-type integrals also in the domain $c^2/(\bar{E}_1 \cup E_2 \cup D)$ possess many properties analogous to the properties of analytic functions. The term "nonanalyticity," introduced by L. A. Aizenberg, cannot serve to characterize their behavior. Thus, it becomes necessary to call the domain $c^2/(\bar{E}_1 \cup E_2 \cup D)$ the domain of Temlyakov quasi-analyticity, understanding by this term the entire set of properties inherent in the integrals F and Φ.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.