ON THE SOLUTION OF CERTAIN NONLINEAR SYSTEMS OF DIFFERENCE EQUATIONS

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Abstract

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ON THE SOLUTION OF CERTAIN NONLINEAR SYSTEMS OF DIFFERENCE EQUATIONS

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The present paper is devoted to the construction of rapidly convergent iterative processes for solving nonlinear systems of difference equations and is conceptually connected with the works ([1–8]), in which a more detailed bibliography can also be found. One of the main results of this paper is the derivation, for difference analogues of the first boundary-value problem in a parallelepiped for strongly elliptic systems in partial derivatives with power nonlinearity, of an iterative method that makes it possible to find the solution of the difference analogue with accuracy ε in $O(N \ln N |\ln \varepsilon|)$ arithmetic operations (N is the total number of unknowns).

1. Let $\omega \equiv \{h\}$ be some set in a Banach space, containing a sequence converging to 0, but $0 \notin \omega$. For each $h \in \omega$ we define: $\dot{H}$ is a finite-dimensional real Hilbert space of dimension $\dot{N}$ with scalar product $(\dot{u}, \dot{v})$ and $\|\dot{u}\| \equiv (\dot{u}, \dot{u})^{1/2}$; $\dot{B}$ is a linear self-adjoint and positive operator mapping $\dot{H}$ into $\dot{H}$; $\dot{H}_B$ is a Hilbert space differing from $\dot{H}$ by the definition of the scalar product:

$$(\dot{u}, \dot{v})_{\dot{H}_B} \equiv (\dot{u}, \dot{v})_{\dot{B}} \equiv (\dot{B}\dot{u}, \dot{v}),$$

$$\|\dot{u}\|_B^2 \equiv (\dot{u}, \dot{u})_B;$$

$\dot{L}$ is some nonlinear operator mapping $\dot{H}$ into $\dot{H}$; $\dot{S}(\dot{u}; r) \equiv \{\dot{v} : \|\dot{v} - \dot{u}\|_B \leq r\}$, $r > 0$; here and in what follows we indicate the dependence on $h \in \omega$ of the concepts used by means of a dot over the corresponding symbols.

Definition 1. We shall say that $\omega, \dot{H}, \dot{L}, \dot{B}$ are connected by the relation $C^0(\dot{u}, r)$ if on $[0, r]$ there exist bounded functions $\delta_0(t)$, $-\delta_1(t)$, nondecreasing as t decreases, such that $\delta_0 > 0$ and, for all $\|\dot{z}\|_B \leq r$, the inequalities

$$\delta_0(\|\dot{z}\|_B)\|\dot{z}\|_B^2 \leq (\dot{L}(\dot{u} + \dot{z}) - \dot{L}(\dot{u}), \dot{z}), \tag{1}$$

$$\|\dot{B}^{-1}(\dot{L}(\dot{u} + \dot{z}) - \dot{L}(\dot{u}))\|_B^2 \leq \delta_1(\|\dot{z}\|_B)\|\dot{z}\|_B^2. \tag{2}$$

Definition 2. We shall say that $\omega, \dot{H}, \dot{B}, \dot{L}$ are connected by the relation $C^1(\dot{u}, r)$ if for any $\|\dot{z}\|_B \le r$ there exists

$$\lim_{\alpha\to 0} \alpha^{-1}(\dot{L}(\dot{u} + (1+\alpha)\dot{z}) - \dot{L}(\dot{u} + \dot{z})) \equiv \dot{L}'(\dot{u} + \dot{z})\dot{z},$$

$\dot{L}'(\dot{u} + \dot{z})$ is a linear operator continuous with respect to $\dot{z}$, and there exist on $[0, r]$ bounded functions $\sigma_0(t)$, $-\sigma_1(t)$, $-\sigma_2(t)$, nondecreasing as t decreases, such that $\sigma_0(r) > 0$ and the inequalities hold:

$$\sigma_0(\|\dot{z}\|_B)\|\dot{z}\|_B^2 \le (\dot{L}_c'(\dot{u} + \dot{z})\dot{z}, \dot{z}) \le \sigma_1(\|\dot{z}\|_B)\|\dot{z}\|_B^2, \tag{3}$$

$$\|\dot{B}^{-1}\dot{L}_k'(\dot{u} + \dot{z})\dot{z}\|_B^2 \le \sigma_2(\|\dot{z}\|_B)\|\dot{z}\|_B^2, \tag{4}$$

where $\dot{L}_c'(\dot{u}+\dot{z})$, $\dot{L}_k'(\dot{u}+\dot{z})$ are, respectively, the symmetric and skew-symmetric parts of the operator $\dot{L}'(\dot{u} + \dot{z})$ in $\dot{H}$.

Theorem 1. Suppose that $\omega, \dot{H}, \dot{B}, \dot{L}$ are connected by the relation $C^0(\dot{u}, r)$ or $C^1(\dot{u}, r)$ with functions $\tilde{\delta}_0(t)$, $\tilde{\delta}_1(t)$, $\tilde{\sigma}_0(t)$, $\tilde{\sigma}_1(t)$, $\tilde{\sigma}_2(t)$, and

$$\dot{B} \equiv \tilde{\dot{B}}(\dot{E} - \dot{T}_M)^{-1}, \tag{5}$$

where $\dot{T}_M$ is a linear self-adjoint operator in the space $\dot{H}_{\tilde{\dot{B}}}$, and $\|\dot{T}_M\|_{\tilde{\dot{B}}} \le q < 1$, $\dot{E}$ is the identity operator. Then $\omega, \dot{H}, \dot{B}, \dot{L}$ are connected

with respect to, respectively, $C^0(\dot{u}, r')$ or $C^1(\dot{u}, r')$, with the functions $\delta_0(t) = \bar{\delta}_0(t')(1-q)$, $\delta_1(t) = \bar{\delta}_1(t')(1+q)^2$, $\sigma_0(t) = \bar{\sigma}_0(t')(1-q)$, $\sigma_1(t) = \bar{\sigma}_1(t')(1+q)$, $\sigma_2(t) = \bar{\sigma}_2(t')(1+q)^2$, $r' \equiv r(1-q)^{-1/2}$, $t' \equiv t(1+q)^{1/2}$.

Theorem 2. Let ω, $\dot{H}$, B, $\dot{L}$ be related by the relation $C^0(\dot{u}, r)$ or $C^1(\dot{u}, r)$, $\|\dot{z}\|_B \le r$, $\gamma > 0$, and $\gamma \le 2(\sigma_0(r) + \sigma_1(r))^{-1}$ in the case $C^1(\dot{u}, r)$. Then

$$\left\|\dot{z} - \gamma\dot{B}^{-1}(\dot{L}(\dot{u} + \dot{z}) - \dot{L}(\dot{u}))\right\|_B \le \rho(\gamma)\|\dot{z}\|_B, \tag{6}$$

where $\rho(\gamma)$ for the case $C^0(\dot{u}, r)$ or $C^1(\dot{u}, r)$ has, respectively, the form

$$\rho(\gamma) \equiv \rho_I(\gamma) = \left(1 - 2\gamma\delta_0(r) + \gamma^2\delta_1(r)\right)^{1/2}, \tag{7}$$

$$\rho(\gamma) \equiv \rho_{II}(\gamma) = \left\{1 - 2\gamma[\sigma_0(r) + \sigma_1(r)] + \gamma^2\left[\sigma_2(r) + \left(\frac{\sigma_0(r) + \sigma_1(r)}{2}\right)^2\right]\right\}^{1/2} + \frac{\sigma_1(r) - \sigma_0(r)}{2}\gamma; \tag{8}$$

$$\min_\gamma \rho_I(\gamma) = \rho_I(\gamma_I) < 1, \qquad \min_\gamma \rho_{II}(\gamma) = \rho_{II}(\gamma_{II}) < 1. \tag{9}$$

It follows from Theorem 2 that, for $\dot{u}^0 \in S(\dot{u}, r)$ and a suitable choice of γ_k, the functions $\dot{u}^{k+1}$ obtained from the iterative process

$$\dot{B}\dot{u}^{k+1} = \dot{B}\dot{u}^k - \gamma_k(\dot{L}(\dot{u}^k) - \dot{L}(\dot{u})) \tag{10}$$

converge to $\dot{u}$.

2. We present some of the studied types of nonlinear systems of difference equations for which the results of Sec. 1 are applicable. Let Ω be a bounded domain in the space $x \equiv (x_1, x_2, \dots, x_n)$; $h_s > 0$ the mesh step in x_s; $h \equiv (h_1, h_2, \dots, h_n)$; $\omega = \{h : \max_s h_s \le h_0,\ h_0 > 0\}$; $i \equiv (i_1, i_2, \dots, i_n)$ a vector with integer components; $x_i \equiv (i_1 h_1, i_2 h_2, \dots, i_{nh} n)$; $\dot{\Omega} \equiv \{x_i : x_i \in \Omega\}$; $\dot{H}$ a Hilbert space of mesh vector-functions $\dot{u} \equiv (\dot{u}_1, \dot{u}_2, \dots, \dot{u}_N)$, defined on $\dot{\Omega}$ and equal to zero outside $\dot{\Omega}$;

$$(\dot{u}, \dot{v}) = h_1 h_2 \cdots h_n \sum_{r=1}^{N} \sum_{x_i \in \dot{\Omega}} \dot{u}_r(x_i) \dot{v}_r(x_i), \qquad \|\dot{u}\| \equiv (\dot{u}, \dot{u})^{1/2}.$$

Let $\{\alpha\}$ be the space of vectors $\alpha \equiv (\alpha_1, \dots, \alpha_n)$, where α_s are natural numbers, $|\alpha| \equiv \sum_{s=1}^{n} \alpha_s$, and let the operators $\dot{B}$ and $\dot{P}$ be defined in $\dot{H}$ by

$$\dot{B}\dot{u} \equiv (\dot{B}_1 \dot{u}_1, \dots, \dot{B}_N \dot{u}_N), \qquad \dot{B}_r \dot{u}_r \equiv (-1)^{m_r} \sum_{s=1}^{n} \bar{\partial}_s^{m_r} \partial_s^{m_r} \dot{u}_r, \tag{11}$$

$$\dot{P}(\dot{u}) \equiv (\dot{P}_1(\dot{u}), \dots, \dot{P}_N(\dot{u})),$$

$$\dot{P}_r(\dot{u}) \equiv \lambda \sum_{|\alpha| \le m_r} (-1)^{|\alpha|} \left[\bar{\partial}^{\alpha} a_{r,\alpha}(x, \partial^{\beta} \dot{u}_l) + \partial^{\alpha} a_{r,\alpha}(x, \bar{\partial}^{\beta} \dot{u}_l) \right], \tag{12}$$

where $\bar{\partial}_s, \partial_s$ are the left and right differences with respect to x_s; $\bar{\partial}^{\alpha} \equiv \bar{\partial}_1^{\alpha_1} \cdots \bar{\partial}_n^{\alpha_n}$; $m_r \ge 1$; $\beta \in \{\alpha\}$; $a_{r,\alpha}$ depend, generally speaking, on x and all $\partial^{\beta} \dot{u}_l (\bar{\partial}^{\beta} \dot{u}_l)$, $|\beta| \le m_l$, $1 \le l \le N$; λ is a number; let $\psi \equiv \{(r, \alpha) : m_r - |\alpha| > n/2,\ 1 \le r \le N\}$, $\psi^* \equiv \{(r, \alpha) : |\alpha| \le m_r,\ (r, \alpha) \notin \psi,\ 1 \le r \le N\}$, and $d_{r,\alpha,l,\beta}(|t^*_{k,\gamma}|, |\tau^*_{k,\gamma}|)$ denote the sum of terms of the form $\prod_{k,\gamma} |t^*_{k,\gamma}|^{q_{k,\gamma}} |\tau^*_{k,\gamma}|^{p_{k,\gamma}}$, where $\gamma \in \{\alpha\}$, $(k, \gamma) \in \psi^*$, and $q_{k,\gamma} \ge 0$, $p_{k,\gamma} \ge 0$ are such that, for $u_k, v_k, z_k \in W_2^{m_k}$, $1 \le k \le N$, the functions $\left(\prod_{k,\gamma} |D^{\gamma} u_k|^{q_{k,\gamma}} |D^{\gamma} v_k|^{p_{k,\gamma}}\right) |D^{\alpha} z_r| \in L_1(\Omega)$, by virtue of the inequalities-

of Hölder and embedding theorems of S. L. Sobolev. Continuous functions of t, τ, monotonically increasing in t, τ, will be denoted by $\chi(t, \tau)$; $\chi(0, 0) = 0$; K_0, K are constants.

Theorem 3. *Let all* $a_{r,\alpha} \in \dot{P}$ *be such that for any* $K_0 > 0$*, when*

$$\max_{(l,\beta)\in\Psi} \max(|t_{l,\beta}|, |\tau_{l,\beta}|) \le K_0$$

there is a $K(K_0)$ *such that*

$$\left|a_{r,\alpha}(x, t_{l,\beta} + \tau_{l,\beta}) - a_{r,\alpha}(x, t_{l,\beta})\right| \le K(K_0) \sum_{l,\beta} |\tau_{l,\beta}|\, d_{r,\alpha,l,\beta}(|t^*_{l,\beta}|, |\tau^*_{l,\beta}|). \quad (13)$$

Then there exist $\chi_1(t,\tau), \chi_2(t,\tau)$ *such that, for* $u \in S(v,r)$*,* $u + z \in S(v,r)$*,*

$$|(\dot{P}(u+z) - \dot{P}(u), z)| \le |\lambda|\chi_1(\|u\|_{\widetilde{B}}, \|z\|_{\widetilde{B}})\|z\|^2_{\widetilde{B}}, \quad (14)$$

$$(\dot{\widetilde{B}}^{-1}(\dot{P}(u+z) - \dot{P}(u)),\ \dot{P}(u+z) - \dot{P}(u)) \le |\lambda|^2\chi_2(\|u\|_{\widetilde{B}}, \|z\|_{\widetilde{B}})\|z\|^2_{\widetilde{B}}. \quad (15)$$

Theorem 4. *Let all derivatives* $\partial a_{r,\alpha}(x, t_{l,\beta})/\partial t_{l,\beta}$ *be continuous in the space* $(x, t_{l,\beta})$ *and, when*

$$\max_{(l,\beta)\in\Psi} \max(|t_{l,\beta}|, |\tau_{l,\beta}|) \le K_0,$$

$$\left|\frac{\partial a_{r,\alpha}}{\partial t_{l,\beta}}(x, t_{l,\beta} + \tau_{l,\beta})\right| \le K(K_0) d_{r,\alpha,l,\beta}(|t^*_{l,\beta}|, |\tau^*_{l,\beta}|). \quad (16)$$

Then there exist $\chi_3(t,\tau), \chi_4(t,\tau)$ *such that, for* $u \in S(v,r)$*,* $u + z \in S(v,r)$*,*

$$|(\dot{P}'_c(u+z)z, z)| \le \chi_3(\|u\|_{\widetilde{B}}, \|z\|_{\widetilde{B}})\|z\|^2_{\widetilde{B}};$$

$$(\dot{\widetilde{B}}^{-1}\dot{P}'_k(u+z)z,\ \dot{P}'_k(u+z)z) \le \chi_4(\|u\|_{\widetilde{B}}, \|z\|_{\widetilde{B}})\|z\|^2_{\widetilde{B}}, \quad (17)$$

where $\dot{P}'(u+z)z \equiv (\dot{P}'_1 z, \dots, \dot{P}'_{Nz})$,

$$\dot{P}'_{rz} \equiv \lambda \sum_{|\alpha|\le m_r} (-1)^{|\alpha|} \left\{ \bar{\partial}^\alpha \sum_{l=1}^{N} \sum_{|\beta|\le m_l} \frac{\partial a_{r,\alpha}}{\partial t_{l,\beta}}(x, \partial^\beta u_l + \partial^\beta z_l)\partial^\beta z_l + \right.$$

$$\left. + \partial^\alpha \sum_{l=1}^{N} \sum_{|\beta|\le m_l} \frac{\partial a_{r,\alpha}}{\partial t_{l,\beta}}(x, \bar{\partial}^\beta u_l + \bar{\partial}^\beta z_l)\bar{\partial}^\beta z_l \right\}, \quad (18)$$

$\dot{P}'(u+z)z$ is the Gâteaux differential for $\dot{P}(u+z)$, while $\dot{P}'_c(u+z)$, $\dot{P}'_k(u+z)$ are the symmetric and skew-symmetric parts in $\dot{H}$ of the operator $\dot{P}'(u+z)$.

Theorem 5. *Let*

$$\dot{\Lambda} u \equiv (\Lambda_1 u, \dots, \Lambda_{Nu}), \qquad \dot{L}(u) \equiv \dot{\Lambda} u + \dot{P}(u), \quad (19)$$

where

$$\Lambda_{ru} \equiv \sum_{l=1}^{N} \sum_{|\alpha|\le m_r} (-1)^{|\alpha|} \left[\bar{\partial}^\alpha \sum_{|\beta|\le m_l} b_{r,l,\alpha,\beta}(x)\partial^\beta u_l + \partial^\alpha \sum_{|\beta|\le m_l} b_{r,l,\alpha,\beta}(x)\bar{\partial}^\beta u_l \right];$$

$\omega, \dot{H}, \dot{\widetilde{B}}, \dot{\Lambda}$ are related by the relation $C^0(u,r)$ $(C^1(u,r))$, and the conditions of Theorem 3 (4) are fulfilled.*

Then, for any $r > 0$, there is a $\lambda_0(r) > 0$ such that, for $|\lambda| \le \lambda_0$, $\omega, \dot{H}, \dot{\widetilde{B}}, \dot{L}$ are also related by the relation $C^0(u,r)$ $(C^1(u,r))$.

Theorem 6. *Let the conditions of Theorem 5 be fulfilled, $|\lambda| \le \lambda_0$, Ω be an n-dimensional parallelepiped, and for solving the equation*

$$L(u) = f \tag{20}$$

the iterative process (10) with

$$\dot{B}u \equiv (\dot{B}_1 u_1, \dots, \dot{B}_{Nu} N), \qquad \dot{B}_l \equiv \dot{\widetilde{B}}_l (E - \dot{T}_{M,l})^{-1}, \qquad \|\dot{T}_{M,l}\|_{\widetilde{B}} \le q < 1, \tag{21}$$

is used

where E is the identity operator, $\dot{T}_{M,l}$ is the error-reduction operator in the variable-directions method (see ([2,3])), $u^0 \in S(\dot{u}, r)$, and γ is defined from (9).

Then, in order to obtain the estimate

$$\|u^{n+1} - \dot{u}\|_{\widetilde{B}} \leqslant \varepsilon \|u^0 - \dot{u}\|_{\widetilde{B}} \tag{22}$$

it is sufficient to expend

$$O\left(|\ln \varepsilon| \frac{|\ln \min_s h_s|}{h_1 h_2 \dots h_n}\right)$$

arithmetic operations; if $n = 2$, all $m_l = 1$, and $T_{M,l}$ corresponds to the iterative process ([8]), then, with the corresponding choice of γ, to obtain (22) it is sufficient to expend

$$O\left(|\ln \varepsilon| \frac{1}{h_1 h_2}\right)$$

operations.

3. The types of nonlinear systems presented in Sec. 2 occur in the numerical solution of many important applied problems, for example nonlinear problems of the theory of elasticity (see ([9])). Difference schemes for them can be chosen so that the potentiality of the differential operator is preserved, and therefore in (4) one may take $\sigma_2(t) = 0$. For such problems γ in Theorem 6 may be chosen on the basis of the method of steepest descent (see ([5])).

Nonpotential difference operators arise, for example, in the approximation of nonlinear problems in the theory of thin plates (see ([10])). Difference schemes for them can be chosen so that $(\dot{P}(u), u) = 0$. This makes it possible to obtain a priori estimates and existence theorems (see, for example, ([11])); for the problem from ([10]) the following system may serve as such a scheme:

$$c_1(\Lambda_1 + \Lambda_2)^2 w_i - \Lambda_1 w_i \Lambda_2 F_i - \Lambda_2 w_i \Lambda_1 F_i + \bar{\partial}_1 \bar{\partial}_2 w_i \bar{\partial}_1 \bar{\partial}_2 F_i + \partial_1 \partial_2 w_i \partial_1 \partial_2 F_i = f_i, \tag{23}$$

$$\begin{aligned} &c_2(\Lambda_1 + \Lambda_2)^2 F_i - \tilde{\partial}_1 \tilde{\partial}_2 w_i (\bar{\partial}_1 \bar{\partial}_2 w_i + \partial_1 \partial_2 w_i) + \Lambda_1 w_{iT} 1 T_{-1} \Lambda_2 w_i + \\ &\quad + \Lambda_2 w_{iT} 2 T_{-2} \Lambda_1 w_i - \frac{1}{2}(\partial_1 \partial_2 w_i - \bar{\partial}_1 \bar{\partial}_2 w_i)(h_2 \tilde{\partial}_1 \Lambda_2 w_i + h_1 \tilde{\partial}_2 \Lambda_1 w_i) - \\ &\quad - \frac{h_1^2 + h_2^2}{2} \Lambda_{1/2} w_i \Lambda_1 \Lambda_2 w_i = 0, \qquad x_i \in \dot{\Omega}, \end{aligned} \tag{24}$$

where $w_i \equiv w(x_i)$, $\Lambda_s \equiv \bar{\partial}_s \partial_s$, $\tilde{\partial}_s \equiv {}^1/_2(\partial_s + \bar{\partial}_s)$, $T_{sw} i \equiv {}^1/_2(w(x_i) + w(x_i + h_{se} s))$, $T_{-s} w_i \equiv {}^1/_2(w(x_i) + w(x_i - h_{se} s))$, $e_1 \equiv (1, 0)$, $e_2 \equiv (0, 1)$,

$$\Lambda_{1/2} w_i \equiv \frac{1}{h_1^2 + h_2^2}(w(x_i + h_1 e_1 + h_2 e_2) - 2w(x_i) + w(x_i - h_1 e_1 - h_2 e_2)).$$

Equations (23), (24) approximate the equations from ([10]) with accuracy $O(h_1^2 + h_2^2)$. The constructions presented are applicable also to more general boundary-value problems.

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Note: Figure translations are in progress. See original paper for figures.