EXPANSION IN EIGENFUNCTIONS OF A SECOND-ORDER DIFFERENTIAL EQUATION WITH OPERATOR COEFFICIENTS

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Abstract

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V. I. GORBACHUK, M. L. GORBACHUK

EXPANSION IN EIGENFUNCTIONS OF A SECOND-ORDER DIFFERENTIAL EQUATION WITH OPERATOR COEFFICIENTS

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The results of G. Weyl ([1]) are known, concerning expansion in eigenfunctions of the self-adjoint boundary-value problem

$$l[y] = y'' + p(t)y = \lambda y \quad (0 \le t < \infty), \qquad y'(0) - hy(0) = 0,$$

where $p(t)$ and h are real. F. S. Rofe-Beketov ([2]) generalized these results to the case where $p(t)$ and h are bounded self-adjoint operators in a separable Hilbert space H.

In the present paper we consider the case where $p(t)$ is an unbounded operator of the form $p(t) = A - q(t)$, where A is a self-adjoint operator in H, semi-bounded below, and $q(t)$ is a bounded self-adjoint operator. This case includes, for example, eigenvalue problems for certain hyperbolic equations.

1. Let H be a separable Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Denote by $L_2(H,(0,b))$ $(0 < b \le \infty)$ the set of all vector-functions $u(t)$ $(0 \le t \le b)$ with values in H such that

$$\int_0^b \|u(t)\|^2\,dt < \infty.$$

As is known, $L_2(H,(0,b))$ is a complete Hilbert space with scalar product

$$(u,v)_b = \int_0^b (u(t),v(t))\,dt \qquad (u,v \in L_2(H,(0,b))).$$

Consider the differential equation

$$l[u] = u'' + Au - q(t)u = \lambda u \tag{1}$$

with the boundary condition

$$u'(0) = 0^*, \tag{2}$$

where $q(t) = q^*(t)$ (* denotes passage to the adjoint operator) is an operator-function, continuous in the uniform operator topology, whose values are bounded operators in H; λ is a complex number; A is a self-adjoint operator in H, semibounded below. Without loss of generality one may assume that $A > 0$ and the operator A^{-1} is bounded. We also assume that the functions $A^{1/2}q(t)A^{-1/2}$ and $Aq(t)A^{-1}$ are strongly continuous in t.

A vector-function $u(t)$ is called a strong solution of equation (1) if $u(t)$, for every t, belongs to $D(A)$ ($D(A)$ is the domain of definition of the operator A), is twice strongly differentiable, and satisfies equation (1).

* The condition (2) is considered for simplicity of exposition. The results are also valid for a boundary condition of the form $u'(0) = Bu(0)$, where B is a bounded self-adjoint operator with the property $BD(A) \subset D(A)$.

On the set $H_+ = D(A)$ introduce the scalar product $(f, g)_+ = (Af, Ag)$. Then H_+ is a complete Hilbert space with respect to $(\cdot,\cdot)_+$, and it may be regarded as a space with positive norm with respect to $H_0 = H$ (see (3)). We denote by H_- the space with negative norm constructed from H_+ and H_0.

A vector function $u(t)$ with values in H will be called a weak solution of equation (1) if $u(t)$ is twice weakly differentiable in H_- (i.e., the scalar function $(u(t), f)$ is twice differentiable for every $f \in H_+$) and

$$\frac{d^2}{dt^2}(u(t), f) + (u(t), Af) - (u, q(t)f) = \lambda(u(t), f) \qquad (f \in H_+).$$

By the method of successive approximations it is not difficult to show that the integral equations

$$\omega_1(t,\lambda) = \cos\sqrt{A-\lambda E}\, t + \int_0^t \frac{\sin\sqrt{A-\lambda E}(t-x)}{\sqrt{A-\lambda E}}\, q(x)\omega_1(x,\lambda)\, dx,$$

$$\omega_2(t,x,\lambda) = \frac{\sin\sqrt{A-\lambda E}(t-x)}{\sqrt{A-\lambda E}} + \int_x^t \frac{\sin\sqrt{A-\lambda E}(t-s)}{\sqrt{A-\lambda E}}\, q(s)\omega_2(s,x,\lambda)\, ds$$

(E is the identity operator in H) have solutions in the class of strongly continuous operator functions. For fixed $t, x \in [0, b]$, the solutions $\omega_1(t,\lambda)$ and $\omega_2(t,x,\lambda)$ are entire functions of λ.

If $f \in D(A)$, $g \in D(A^{1/2})$, then $\omega_1(t,\lambda)f$ and $\omega_2(t,0,\lambda)g$ are strong solutions of equation (1), satisfying the initial data

$$\omega_1(0,\lambda)f = f, \qquad \omega_1'(0,\lambda)f = 0; \qquad \omega_2(0,0,\lambda)g = 0, \qquad \omega_2'(0,0,\lambda)g = g.$$

If $f, g \in H$, then $\omega_1(t,\lambda)f$ and $\omega_2(t,0,\lambda)g$ are weak solutions of this equation.

Theorem 1. The Cauchy problem $u(0) = f,\ u'(0) = g\ (f, g \in H)$ for equation (1) has a unique weak solution. This solution has the form

$$u(t,\lambda) = \omega_1(t,\lambda)f + \omega_2(t,0,\lambda)g.$$

2. Denote by D' the set of all functions $u(t) \in L_2(H,(0,b))$ that are twice strongly differentiable in H, for each fixed $t \in [0,b]$ belong to $D(A)$, satisfy condition (2), and are such that $l[u] \in L_2(H,(0,b))$. On D' define the operator L': $L'u = l[u]$. Denote also by D_0' the set of functions $u(t) \in D'$ finite in a neighborhood of the point b, and by L_0' the restriction of L' to D_0'. The operator L_0' is Hermitian and $L' \subset L_0'^*$. Let L_0 be the closure of L_0'.

Theorem 2. $D(L_0^*)$ consists of functions $u(t) \in L_2(H,(0,b))$ having two weak derivatives in H_- and such that $u'(0) = 0$. If $L_0^* u = u^*$, then

$$u(t) = \omega_1(t,0)f + \int_0^t \omega_2(t,x,0)u^*(x)\,dx \qquad (f = u(0) \in H). \tag{3}$$

In the case $b < \infty$, the operator L_0^* coincides with the closure of the operator L' in $L_2(H,(0,b))$, and $D(L_0)$ consists of functions $u(t) \in L_2(H,(0,b))$, twice weakly differentiable in H_-, and such that $u'(0) = u(b) = u'(b) = 0$.

3. Denote by $\mathfrak{N}_\lambda$ the defect subspace of the operator L_0. The subspace $\mathfrak{N}_\lambda$ consists of vectors of the form $\omega_1(t,\overline{\lambda})f$, where $f \in H$ is such that

$$\int_0^b \|\omega_1(t,\overline{\lambda})f\|^2\,dt < \infty.$$

In what follows we shall assume that $b < \infty$. In this case L_0 has infinite defect numbers.

Put

$$I_\lambda = I_{\lambda,b} = \int_0^b \omega_1^*(t,\lambda)\omega_1(t,\lambda)\,dt.$$

Lemma 1. The operator I_λ is invertible for every complex λ.

Let $\widetilde{L}$ be a self-adjoint extension in $L_2(H,(0,b))$ of the operator L_0. Then $D(\widetilde{L})$ consists of elements of the form

$$y = u + \omega_1(\cdot, \bar{\lambda}_0) I_{\lambda_0}^{-1/2} U I_{\lambda_0}^{1/2} f - \omega_1(\cdot, \lambda_0) f, \tag{4}$$

where $u \in D(L_0)$; $f \in H$; U is a unitary operator in H; λ_0 is a fixed nonreal number ($\operatorname{Im}\lambda_0 > 0$). Conversely, if U is a unitary operator in H, then formula (4) defines the domain of a certain self-adjoint extension $\widetilde{L}$ of the operator L_0 in $L_2(H,(0,b))$.

Theorem 3. A vector-function $y(t) \in D' \cap D(\widetilde{L})$ if and only if it satisfies the boundary condition

$$[\omega_1^*(b,\lambda_0) - I_{\lambda_0}^{1/2} U I_{\bar{\lambda}_0}^{-1/2} \omega_1^*(b,\bar{\lambda}_0)] y'(b) -$$

$$-[\omega_1'^*(b,\lambda_0) - I_{\lambda_0}^{1/2} U I_{\bar{\lambda}_0}^{-1/2} \omega_1'^*(b,\bar{\lambda}_0)] y(b) = 0. \tag{5}$$

Between the set of all self-adjoint extensions of L_0 in $L_2(H,(0,b))$ and the set of all unitary operators U in H there exists a one-to-one correspondence.

4. We shall say that an operator-function $M(z)$ in H belongs to the class R if it is analytic in the upper half-plane and $\operatorname{Im} M(z) = [M(z) - M^*(z)]/2i \geq 0$ for $\operatorname{Im} z > 0$. An operator-function $M(z)$ of class R is represented uniquely in the form

$$M(z) = P + Qz + \int_{-\infty}^{\infty} \frac{1+tz}{t-z}\, d\sigma(t), \tag{6}$$

where P is a bounded self-adjoint operator and Q is a bounded positive operator; $\sigma(t)$ ($\sigma(0) = 0$, $\sigma(t-0) = \sigma(t)$ in the strong sense) is a monotonically increasing operator function such that $\sigma(+\infty)$ is a bounded operator.

Theorem 4. There exists an operator-function $M(z) \in R$ such that

$$\chi(b,\lambda) = \omega_2(b,0,\lambda) - \omega_1(b,\lambda) M(\lambda) = 0.$$

Denote

$$G(t,s,\lambda) = \begin{cases} \omega_1(t,\lambda)\chi^*(s,\bar{\lambda}), & \text{for } 0 \leq t \leq s \leq b, \\ \chi(t,\lambda)\omega_1^*(s,\bar{\lambda}), & \text{for } 0 \leq s \leq t \leq b, \end{cases}$$

where $\chi(t,\lambda) = \omega_2(t,0,\lambda) - \omega_1(t,\lambda) M(\lambda)$. The function $G(t,s,\lambda)$ has the following properties:

1) for fixed s and t, $G(t, s, \lambda)$ is an analytic function of λ ($\operatorname{Im} \lambda \neq 0$);

2) for fixed λ, $G(t, s, \lambda)$ $(0 \leq t, s \leq b)$ is a strongly continuous function jointly in the variables t and s;

3) for fixed s $(0 \leq s \leq b)$, the vector-function $G(t, s, \lambda) f,\ f \in H$, has a weak derivative with respect to t in H in each of the intervals $[0, s)$ and $(s, b]$, and, at $t = s$,

$$\frac{\partial}{\partial t} G(s+0, s, \lambda) f - \frac{\partial}{\partial t} G(s-0, s, \lambda) f = f;$$

4) in each of the intervals $[0, s)$ and $(s, b]$, $G(t, s, \lambda) f,\ f \in H$, as a function of t, is a weak solution of equation (1), satisfying the conditions $u'(0) = 0,\ u(b) = 0$.

Let L_1 be the closure in $L_2(H, (0, b))$ of the operator L_1', defined on the set of functions $y(t) \in D'$ satisfying the conditions $y'(0) = 0,\ y(b) = 0$, by means of the differential expression $l[y]$. L_1 is a self-adjoint extension of the operator L_0, and its resolvent, by virtue of properties 1)–4), admits the integral representation

$$R_\lambda y = (L_1 - \lambda E)^{-1} y = \int_0^b G(\cdot, s, \lambda) y(s)\, ds. \tag{7}$$

5. Introduce the ω_1-transform of a function $y(t) \in L_2(H, (0, b))$:

$$\tilde{y}(\lambda) = \int_0^b \omega_1^*(t, \bar{\lambda}) y(t)\, dt.$$

An operator function $\rho(\lambda)$ $(\rho(\lambda - 0) = \rho(\lambda),\ \rho(0) = 0)$ will be called a distribution function if $\rho(\Delta) = \rho(\lambda + \Delta) - \rho(\lambda)$ is a positive bounded operator in H.

Theorem 5. *There exists an operator distribution function $\rho(\lambda)$ $(-\infty < \lambda < \infty)$ such that, for any $y(t), z(t) \in L_2(H, (0, b))$,*

$$\int_0^b (y(t), z(t))\, dt = \int_{-\infty}^{\infty} (d\rho(\lambda)\, \tilde{y}(\lambda), \tilde{z}(\lambda)) \tag{8}$$

$(d\rho(\lambda) = (1+\lambda^2) d\sigma(\lambda)$, where $\sigma(\lambda)$ is the function from the representation (6), uniquely determined by $M(z)$

An operator function $\rho(\lambda)$ for which (8) holds will be called the spectral function of problem (1)–(2) on the interval $[0, b]$.

6. Consider the case when $b = \infty$. If $\beta_1 < \beta_2 < \infty$, then $0 < I_{\lambda,\beta_1} < I_{\lambda,\beta_2}$; on the basis of (3), $I^{-1}_{\lambda,\beta_1} \geq I^{-1}_{\lambda,\beta_2}$, i.e. $I^{-1}_{\lambda,\beta}$ $(0 \leq \beta < \infty)$ is a monotonically decreasing operator function in H. Therefore, as $\beta \to \infty$, it converges to a positive bounded operator Γ_λ. Similarly to how this is done in [4], one can prove the following theorems.

Theorem 6. *For each* λ $(\operatorname{Im}\lambda \neq 0)$ *there exists an operator solution* $x(t,\lambda)$ *of equation (1) such that, for all* $f \in H$,

$$\int_0^\infty \|x(t,\lambda)f\|^2 dt < \infty.$$

Theorem 7. *The dimension of the orthogonal complement to the null manifold of the operator* Γ_λ $(\operatorname{Im}\lambda \neq 0)$ *coincides with the dimension of the defect subspace* $\mathfrak{N}_\lambda$ *of the operator* L_0*, and hence is the same for all* λ *from a connected component of the regularity field of the operator* L_0.

Using Theorem 5 for arbitrary $\beta < \infty$, by the method of stretching the interval (see [5]) we arrive at the assertion.

Theorem 8. *There exists a spectral function* $\rho(\lambda)$ *of problem (1)–(2) on the half-axis* $(0,\infty)$*, i.e. one such that, for any* $y, z \in L_2(H,(0,b))$,

$$\int_0^\infty (y(t), z(t))\, dt = \int_{-\infty}^\infty (d\rho(\lambda)\, \tilde{y}(\lambda), \tilde{z}(\lambda)) \left(\int_{-\infty}^\infty \frac{(d\rho(\lambda)f, f)}{1+\lambda^2} < \infty \right).$$

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Institute of Mathematics
Academy of Sciences of the Ukrainian SSR

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