A Game Variant of the Optimal Stopping Problem

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Abstract

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A Game Variant of the Optimal Stopping Problem

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1. Let an increasing sequence of σ-algebras $\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \cdots \subseteq \mathscr{F}_n \subseteq \cdots$ be given on a space Ω, and let functions $x_n(\omega)$, $\varphi_n(\omega)$, measurable with respect to $\mathscr{F}_n$ $(n = 0, 1, 2, \ldots)$, be given. The process may be stopped by the first player at those times n when $\varphi_n > 0$, and by the second player at those times when $\varphi_n < 0$. If stopping is carried out at time n, then the second player receives x_n from the first. In this game the strategies of the first player are Markov times σ subject to the condition: $\varphi_\sigma > 0$ when $\sigma < \infty$; the strategies of the second player are Markov times τ, for which $\varphi_\tau < 0$ when $\tau < \infty$. (A function $\tau(\omega)$ taking the values $0, 1, \ldots, n, \ldots$ and the value $+\infty$ is called a Markov time if, for every finite n, $\{\tau = n\} \in \mathscr{F}_n$.) Let P be a probability measure on some σ-algebra $\mathscr{F}$ containing $\mathscr{F}_n$ for all n, and let $M\xi$ denote the integral of ξ with respect to P. The mean payoff corresponding to the strategies σ, τ is equal to $Mx_{\sigma\wedge\tau}$.*

We shall assume that the following condition is satisfied.

Condition A. $M\{\sup_n |x_n|\} < \infty$.

Under this condition the existence of the value of the game is proved, ϵ-optimal strategies are constructed, and some conditions for the existence of optimal strategies are derived.

If $\varphi_n < 0$ everywhere, then we arrive at the optimal stopping problem considered by Snell (^{1}) and by other authors. Of special interest is also the case $x_n = g(X_n)$, $\varphi_n = \Phi(X_n)$, where $(X_t, \mathscr{F}_t, P_x)$ is some Markov chain. This case has been studied independently by E. B. Fried.

2. Suppose the game begins at time n. Then only strategies $\sigma \geq n$ and $\tau \geq n$ are possible. (Throughout the paper it is understood that $\varphi_\sigma > 0$ when $\sigma < \infty$, and $\varphi_\tau < 0$ when $\tau < \infty$.) The corresponding mean payoff is $g(n, \sigma, \tau) = M(x_{\sigma\wedge\tau} \mid \mathscr{F}_n)$.

Let ξ_α be an arbitrary family of measurable functions. We agree to denote by $\operatorname{Sup} \xi_\alpha$ a measurable function ξ satisfying the conditions: a) for every α, $\xi_\alpha \leq \xi$ (a.s.)**; b) if $\xi_\alpha \leq \eta$ (a.s.) for every α, then $\xi \leq \eta$ (a.s.). It is proved (see

(^{1},^{2})) that such a function ξ exists and is defined uniquely up to an a.s. zero summand, and that one can choose a sequence of values α_m so that $\operatorname{Sup}\xi_\alpha = \sup_m \xi_{\alpha_m}$ (a.s.). Put $\operatorname{Inf}\xi_\alpha = -\sup(-\xi_\alpha)$.

The upper and lower values of the game are defined by the formulas

$$\bar{w}_n = \operatorname{Inf}_{\sigma\geq n}\operatorname{Sup}_{\tau\geq n} g(n,\sigma,\tau), \qquad \underline{w}_n = \operatorname{Sup}_{\tau\geq n}\operatorname{Inf}_{\sigma\geq n} g(n,\sigma,\tau). \tag{1}$$

The main results of the paper are contained in the following two theorems.

Theorem 1. *Almost surely* $\underline{w}_n = \bar{w}_n$, *and the value of the game* $w = w_n = \bar{w}_n$ *satisfies the equation*

$$w_n = \begin{cases} M(w_{n+1} \mid \mathcal{F}_n) \wedge x_n, & (\text{a.s. } \varphi_n > 0) * **, \\ M(w_{n+1} \mid \mathcal{F}_n) \vee x_n, & (\text{a.s. } \varphi_n < 0), \\ M(w_{n+1} \mid \mathcal{F}_n), & (\text{a.s. } \varphi_n = 0). \end{cases} \tag{2}$$

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* We denote the smaller of two numbers a, b by $a \wedge b$, and the larger by $a \vee b$. The value x_∞ is, by definition, equal to zero.

** "(a.s.)" means almost surely, i.e. for all ω except for a set of measure zero.

*** "(a.s. A)" means for almost all $\omega \in A$.

For any $\varepsilon > 0$ the strategies

$$\begin{aligned} \sigma_\varepsilon^n &= \inf\{t : t \geq n,\ \varphi_t > 0,\ x_t \leq w_t + \varepsilon\}, \\ \tau_\varepsilon^n &= \inf\{t : t \geq n,\ \varphi_t < 0,\ x_t \geq w_t - \varepsilon\} \end{aligned} \tag{3}$$

are ε-optimal in the sense that for any $\sigma \geq n$, $\tau \geq n$

$$g(n,\sigma,\tau_\varepsilon^n) + \varepsilon \geq w_n \geq g(n,\sigma_\varepsilon^n,\tau) - \varepsilon \quad (\text{a.s.}). \tag{4}$$

If $\tau_0^n < +\infty$ (a.s.) ($\sigma_0^n < +\infty$ (a.s.)), then the first (respectively the second) of inequalities (4) remains valid for $\varepsilon = 0$.

Theorem 2. Let

$$s_n = \operatorname*{Inf}_{\sigma\geq n} M(x_\sigma \mid \mathcal{F}_n), \qquad S_n = \operatorname*{Sup}_{\tau\geq n} M(x_\tau \mid \mathcal{F}_n). \tag{5}$$

Then S_n is the smallest supermartingale satisfying the condition

$$S_n \geq x_n \ (\text{a.s. } \varphi_n < 0), \qquad S_n \geq 0 \ (\text{a.s.}); \tag{6}$$

s_n is the largest submartingale satisfying the condition

$$s_n \leq x_n \ (\text{a.s. } \varphi_n > 0), \qquad s_n \leq 0 \ (\text{a.s.}). \tag{7}$$

If with probability 1 there exists $m \geq n$ for which $S_m = x_m$, $\varphi_m < 0$ ($s_m = x_m$, $\varphi_m > 0$), then the first (respectively the second) of inequalities (4) holds for $\varepsilon = 0$.

3. Put

$$\bar{g}(n, \sigma) = \operatorname*{Sup}_{\tau \geq n} g(n, \sigma, \tau). \tag{8}$$

Obviously,

$$\bar{w}_n = \operatorname*{Inf}_{\sigma \geq n} \bar{g}(n, \sigma). \tag{9}$$

Note that

$$\begin{gathered} \bar{w}_n \leq \bar{g}(n, n) = x_n \quad (\text{a.s. } \varphi_n > 0), \\ \bar{w}_n \geq \operatorname*{Inf}_{\delta \geq n} g(n, \sigma, n) = x_n \quad (\text{a.s. } \varphi_n < 0). \end{gathered} \tag{10}$$

Let $\sigma \geq n$, $\tau \geq n$. Put $\sigma' = \sigma \vee (n+1)$, $\tau' = \tau \vee (n+1)$. It is not difficult to verify that

$$g(n, \sigma, \tau) = M\{g(n+1, \sigma', \tau') \mid \mathcal{F}_n\} \quad (\text{a.s. } \sigma > n, \ \tau > n); \tag{11}$$

$$g(n, \sigma, \tau) = x_n \quad (\text{a.s. } \{\sigma = n\} \cup \{\tau = n\}). \tag{12}$$

4. For each n one can choose a sequence of Markov times $\sigma_m \geq n+1$ such that $\bar{w}_{n+1} = \inf_m \bar{g}(n+1, \sigma_m)$ (a.s.). Fix $\varepsilon > 0$ and denote by ν the least value of m for which

$$\bar{g}(n+1, \sigma_m) \leq \bar{w}_{n+1} + \varepsilon. \tag{13}$$

It is easy to see that σ_ν is a Markov time and $\sigma_\nu \geq n+1$. By virtue of (8) and (13), for any $\tau \geq n+1$ and any m

$$g(n+1, \sigma_m, \tau) \leq \bar{g}(n+1, \sigma_m) \leq \bar{w}_{n+1} + \varepsilon \quad (\text{a.s. } \nu = m).$$

Therefore

$$g(n+1,\sigma_\nu,\tau) = M(x_{\sigma_\nu\wedge\tau} \mid \mathcal{F}_{n+1}) = \sum_m M\{\chi_{\nu=m} x_{\sigma_m\wedge\tau} \mid \mathcal{F}_{n+1}\} =$$

$$= \sum_m \chi_{\nu=m} g(n+1,\sigma_m,\tau) \le \bar{w}_{n+1} + \varepsilon \quad \text{(a.s.)}.$$

By virtue of (11)

$$g(n,\sigma_v,\tau) = M\{g(n+1,\sigma_v,\tau') \mid \mathcal{F}_n\} \le M(\overline{w}_{n+1} \mid \mathcal{F}_n) + \varepsilon \quad \text{(a.s. } \tau > n). \tag{14}$$

According to (12)

$$g(n,\sigma_v,\tau) = x_n \quad \text{(a.s. } \tau = n). \tag{15}$$

Since $\{\varphi_n \ge 0\} \subseteq \{\tau > n\}$, it follows from (14) that

$$\overline{g}(n,\sigma_v) \le M\{\overline{w}_{n+1} \mid \mathcal{F}_n\} + \varepsilon \quad \text{(a.s. } \varphi_n \ge 0). \tag{16}$$

Further, from (14) and (15),

$$\overline{g}(n,\sigma_v) \le x_n \vee M(\overline{w}_{n+1} \mid \mathcal{F}_n) + \varepsilon \quad \text{(a.s.)}. \tag{17}$$

Taking (9) into account, we conclude from (16) and (17) that

$$\overline{w}_n \le M(\overline{w}_{n+1} \mid \mathcal{F}_n) \quad \text{(a.s. } \varphi_n \ge 0), \tag{18}$$

$$\overline{w}_n \le x_n \vee M(\overline{w}_{n+1} \mid \mathcal{F}_n) \quad \text{(a.s.)}. \tag{19}$$

5. Let $\sigma \ge n$. By virtue of (8) one can choose a sequence of Markov times $\tau_m \ge n+1$ such that

$$\overline{g}(n+1,\sigma') = \sup g(n+1,\sigma',\tau_m).$$

Let v be the smallest value of m for which

$$g(n+1,\sigma',\tau_m) \ge \overline{g}(n+1,\sigma') - \varepsilon.$$

Then τ_v is a Markov time, $\tau_v \ge n+1$, and

$$g(n+1,\sigma',\tau_v) \ge \overline{g}(n+1,\sigma') - \varepsilon$$

(a.s.). By virtue of (8), (11), and (9),

$$\overline{g}(n,\sigma) \geq g(n,\sigma,\tau_v) = M\{g(n+1,\sigma',\tau_v) \mid \mathcal{F}_n\} \geq$$

$$\geq M\{\overline{g}(n+1,\sigma') \mid \mathcal{F}_n\} - \varepsilon \geq M(\overline{w}_{n+1} \mid \mathcal{F}_n) - \varepsilon \quad (\text{a.s. } \sigma > n). \tag{20}$$

By virtue of (12), $g(n,\sigma,\tau_v) = x_n$ (a.s. $\sigma = n$). Therefore

$$\overline{g}(n,\sigma) \geq x_n \wedge M(\overline{w}_{n+1} \mid \mathcal{F}_n) - \varepsilon \quad (\text{a.s.}). \tag{21}$$

Since $\{\varphi_n \leq 0\} \subseteq \{\sigma > n\}$, according to (20)

$$\overline{w}_n \geq M(\overline{w}_{n+1} \mid \mathcal{F}_n) - \varepsilon \quad (\text{a.s. } \varphi_n \leq 0). \tag{22}$$

In view of the arbitrariness of $\varepsilon > 0$, from (21) and (22) we have

$$\overline{w}_n \geq M(\overline{w}_{n+1} \mid \mathcal{F}_n) \quad (\text{a.s. } \varphi_n \leq 0); \tag{23}$$

$$\overline{w}_n \geq x_n \wedge M(\overline{w}_{n+1} \mid \mathcal{F}_n) \quad (\text{a.s.}). \tag{24}$$

6. From (18) and (23) it follows that

$$\overline{w}_n = M(\overline{w}_{n+1} \mid \mathcal{F}_n)$$

(a.s. $\varphi_n = 0$). From (9), (18), and (24) we have

$$\overline{w}_n = x_n \wedge M(\overline{w}_{n+1} \mid \mathcal{F}_n)$$

(a.s. $\varphi_n > 0$). Finally, from (10), (23), and (19)

$$\overline{w}_n = x_n \vee M(\overline{w}_{n+1} \mid \mathcal{F}_n)$$

(a.s. $\varphi_n < 0$). Thus, $\overline{w}_n$ satisfies equation (2).

When x_n is replaced by $-x_n$ and φ_n by $-\varphi_n$, the functions $\underline{w}_n$ pass into $-\overline{w}_n$. Therefore $\underline{w}_n$ also satisfies equation (2).

7. Fix arbitrary $\varepsilon > 0$ and n, and put

$$\sigma_\varepsilon = \inf\{t : t \geq n,\ \varphi_t > 0,\ x_t \leq \underline{w}_t + \varepsilon\},$$

$$\tau_\varepsilon = \inf\{t : t \geq n,\ \varphi_t < 0,\ x_t \geq \overline{w}_t - \varepsilon\} \tag{25}$$

(if the set is empty, we take its lower bound to be $+\infty$). Let

$$\Lambda_n = \{t : t \geq n,\ \varphi_t < 0\},$$

and let $\tilde{x}_n$ be the exact upper bound of the set x_t $(t \in \Lambda_n)$, augmented by zero. Note that for $m < n$

$$\overline{w}_n \le \sup_{\tau \ge n} g(n, \infty, \tau) \le M(\tilde{x}_n \mid \mathcal{F}_n) \le M(\tilde{x}_m \mid \mathcal{F}_n) \quad \text{(a.s.)}. \tag{26}$$

Therefore

$$\overline{\lim_{n \to \infty}}\, \overline{w}_n \le M(\tilde{x}_m \mid \mathcal{F}_\infty) = \tilde{x}_m \quad \text{(a.s.)}, \tag{27}$$

where $\mathcal{F}_\infty$ is the minimal σ-algebra containing $\mathcal{F}_n$ for all n.

It follows from (27) that

$$\overline{\lim_{n \to \infty}}\, \bar{w}_n \le U \vee 0, \tag{28}$$

where $U = 0$ if Λ_0 is finite, and U is the upper limit of x_t as $t \to \infty$ over the set Λ_0, if Λ_0 is infinite.

Put $v_t = \bar{w}_{\tau_\varepsilon \wedge t}$. By virtue of (2), $M(\bar{w}_{t+1} \mid \mathcal{F}_t) \ge \bar{w}_t$ (a.s. $t < \tau_\varepsilon$). Therefore, for $t \ge n$,

$$M(v_{t+1} \mid \mathcal{F}_t) = \chi_{\tau_\varepsilon > t} M(\bar{w}_{t+1} \mid \mathcal{F}_t) + \sum_{m=n}^{t} \chi_{\tau_\varepsilon = m} \bar{w}_m \ge$$

$$\ge \chi_{\tau_\varepsilon > t} \bar{w}_t + \chi_{\tau_\varepsilon \le t} \bar{w}_{\tau_\varepsilon} = v_t \quad \text{(a.s.)}.$$

Thus $(v_t, \mathcal{F}_t)$ is a submartingale, and for any $\sigma \ge n \,{}^\circ a_1 W \wedge {}^\circ a \le (w_\sigma \mid n = v_n = \bar{w}_n$ (a.s.). By condition A, Fatou' s lemma is applicable to the sequence $v_{\sigma \wedge t}$. Therefore

$$M \left\{ \overline{\lim_{t \to \infty}}\, v_{\sigma \wedge t} \mid \mathcal{F}_n \right\} \ge \overline{\lim_{t \to \infty}}\, M(v_{\sigma \wedge t} \mid \mathcal{F}_n) \ge \bar{w}_n \quad \text{(a.s.)}. \tag{29}$$

Let us note that

$$\overline{\lim_{t \to \infty}}\, v_{\sigma \wedge t} = \overline{\lim_{t \to \infty}}\, \bar{w}_{\tau_\varepsilon \wedge \sigma \wedge t} = \bar{w}_{\tau_\varepsilon \wedge \sigma} \quad \text{when } \tau_\varepsilon \wedge \sigma < \infty. \tag{30}$$

On the other hand, by virtue of (28), if $\tau_\varepsilon = \infty$, then $U \le 0$ (a.s.) and

$$\overline{\lim_{t \to \infty}}\, v_{\sigma \wedge t} \le \overline{\lim_{n \to \infty}}\, \bar{w}_n \le 0 \quad (\text{a.s. } \tau_\varepsilon = \infty). \tag{31}$$

From (29), (30), and (31) we have

$$M(\bar{w}_{\tau_\varepsilon \wedge \sigma} \mid \mathcal{F}_n) \geq \bar{w}_n \quad \text{(a.s.)} \tag{32}$$

(where we put $\bar{w}_\infty = 0$). If $\tau_0 < \infty$ (a.s.), then (32) also holds for $\varepsilon = 0$.

According to (25), $x_{\tau_\varepsilon} = \bar{w}_{\tau_\varepsilon} - \varepsilon$, and by virtue of (2) $x_\sigma \geq \bar{w}_\sigma$ (a.s.) (for $\varphi_\sigma > 0$). Therefore $x_{\sigma \wedge \tau_\varepsilon} \geq \bar{w}_{\sigma \wedge \tau_\varepsilon} - \varepsilon$ (a.s.), and by virtue of (32)

$$\bar{w}_n \leq M(x_{\sigma \wedge \tau_\varepsilon} \mid \mathcal{F}_n) + \varepsilon = g(n, \sigma, \tau_\varepsilon) + \varepsilon \quad \text{(a.s.)}. \tag{33}$$

Replacing x_n by $-x_n$ and φ_n by $-\varphi_n$, we obtain that for any $\tau \geq n$ $w_n \geq g(n, \sigma_\varepsilon, \tau) - \varepsilon$ (a.s.). It is easy to see that $\bar{w}_n \geq w_n$ (a.s.). Consequently, for any $\sigma \geq n$ and $\tau \geq n$,

$$g(n, \sigma_\varepsilon, \tau) - \varepsilon \leq w_n \leq \bar{w}_n \leq g(n, \sigma, \tau_\varepsilon) + \varepsilon \quad \text{(a.s.)}. \tag{34}$$

Putting $\sigma = \sigma_\varepsilon$, $\tau = \tau_\varepsilon$, we find that $0 \leq \bar{w}_n - w_n \leq 2\varepsilon$ (a.s.). Therefore $w_n = \bar{w}_n$ (a.s.), and (4) follows from (34). Theorem 1 is proved.

Theorem 2 is proved without difficulty, and we omit its proof.

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Note: Figure translations are in progress. See original paper for figures.

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