ON THE STABILITY OF BASES OF LOCALLY CONVEX SPACES

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MATHEMATICS

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ON THE STABILITY OF BASES OF LOCALLY CONVEX SPACES

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1. In the present note we give some criteria for the stability of bases (as well as defective bases) in locally convex spaces. In doing so, we are able to indicate a complete characterization of certain classes of bases in terms of the perturbations admissible for them. Some applications are also given of the stability criteria obtained to spaces of functions analytic in a disk.

Everywhere below, E denotes a locally convex space, $\{V_\alpha\}$ $(\alpha \in \mathfrak{A})$ a fundamental system of absolutely convex closed neighborhoods of zero, and $|x|_\alpha$ $(\alpha \in \mathfrak{A})$ the corresponding system of seminorms.

A sequence $\{x_j\}_1^\infty$ of vectors of the space E is called its **basis** if every vector $x \in E$ is represented uniquely in the form of a series

$$x = \sum_{j=1}^{\infty} a_j x_j.$$

By $\{f_j\}_1^\infty$ we shall denote the sequence of functionals defined by the equalities $\langle x, f_j \rangle = a_j$ $(j = 1, 2, \ldots;\ x \in E)$.

A basis $\{x_j\}_1^\infty$ is called **equicontinuous** (respectively **absolute**) if for every index $\beta \in \mathfrak{A}$ there is an index $\varphi(\beta) \in \mathfrak{A}$ such that

$$|\langle x, f_j \rangle|\, |x_j|_\beta \le |x|_{\varphi(\beta)}, \qquad j = 1, 2, \ldots$$

(respectively

$$\sum_{j=1}^{\infty} |\langle x, f_j \rangle|\, |x_j|_\beta \le |x|_{\varphi(\beta)}$$

) for every $x \in E$.

A basis $\{x_j\}_1^\infty$ will be called a **Bessel basis** if for every index $\beta \in \mathfrak{A}$ there is an index $\varphi(\beta) \in \mathfrak{A}$ such that

$$\sum_{j=1}^{\infty} |\langle x, f_j \rangle|^2 |x_j|_\beta^2 \leq |x|_{\varphi(\beta)}^2$$

for every $x \in E$.

As is known, every basis of a barrelled space is equicontinuous, and every equicontinuous basis of a nuclear space is absolute (see, for example, (1)). It is not difficult to show that in a barrelled space E, whose topology is defined by a family of Hermitian forms $(x, y)_\alpha$ (i.e. $|x|_\alpha = \sqrt{(x, x)_\alpha}$), every unconditional basis is a Bessel basis.

A sequence of vectors $\{y_j\}_1^\infty$ is called **ω-linearly independent** if the equality

$$\sum_{j=1}^{\infty} c_j y_j = 0$$

is possible only when $c_j = 0$ $(j = 1, 2, \ldots)$. Two bases $\{x_j\}_1^\infty$ and $\{y_j\}_1^\infty$ are called **equiva-**

lent, if there exists an isomorphism A of the space E onto itself such that $Ax_j = y_j$ $(j = 1, 2, \ldots)$.

2. To simplify the formulations, we shall henceforth assume that the semi-norms $\{|\cdot|_\alpha\}$ $(\alpha \in \mathfrak{A})$ are norms (see Remark 2 below).

Theorem 1. Let $\{x_j\}_1^\infty$ be a basis of the complete* space E, satisfying one of the following conditions:

I. $\{x_j\}_1^\infty$ is an equicontinuously continuous basis.

II. $\{x_j\}_1^\infty$ is an absolute basis.

III. $\{x_j\}_1^\infty$ is a Bessel basis.

Then for any index $\beta \in \mathfrak{A}$ there exists an index $\gamma \in \mathfrak{A}$ such that every sequence $\{y_j\}_1^\infty$ satisfying, respectively, one of the following conditions:

$$1) \quad \sum_{j=1}^{\infty} \frac{|x_j - y_j|_\alpha}{|x_j|_\beta} < \infty \quad (\alpha \in \mathfrak{A}), \qquad \sum_{j=1}^{\infty} \frac{|x_j - y_j|_\gamma}{|x_j|_\beta} < 1;$$

$$2) \quad \text{the set } \left\{ \frac{x_j - y_j}{|x_j|_\beta} \right\}_1^\infty \text{ is bounded and } \frac{x_j - y_j}{|x_j|_\beta} \in V_\gamma \ (j = 1, 2, \ldots);$$

$$3) \quad \sum_{j=1}^{\infty} \frac{|x_j - y_j|_\alpha^2}{|x_j|_\beta^2} < \infty \quad (\alpha \in \mathfrak{A}), \qquad \sum_{j=1}^{\infty} \frac{|x_j - y_j|_\gamma^2}{|x_j|_\beta^2} < 1,$$

is a basis of E, equivalent to $\{x_j\}_1^\infty$.

Remark 1. If the system of neighborhoods of zero $\{V_{\varphi(\alpha)}\}$ $(\alpha \in \mathfrak{A})$ is fundamental, then Theorem 1 remains valid for every sequence $\{y_j\}_1^\infty$ satisfying, respectively, one of the following conditions:

$$1^\circ. \quad \sum_{j=1}^{\infty} \frac{|x_j - y_j|_{\varphi(\alpha)}}{|x_j|_\alpha} \le q_\alpha \quad (q_\alpha < 1;\ \alpha \in \mathfrak{A});$$

$$2^\circ. \quad |x_j - y_j|_{\varphi(\alpha)} \le q_\alpha |x_j|_\alpha \quad (q_\alpha < 1;\ \alpha \in \mathfrak{A};\ j = 1, 2, \ldots);$$

$$3^\circ. \quad \sum_{j=1}^{\infty} \frac{|x_j - y_j|^2_{\varphi(\alpha)}}{|x_j|^2_\alpha} \le q_\alpha \quad (q_\alpha < 1;\ \alpha \in \mathfrak{A}).$$

Theorem 1 admits the following converse.

Theorem 2. Let $\{x_j\}_1^\infty$ be a basis of the space E, and suppose that for each index $\beta \in \mathfrak{A}$ there exists an index $\gamma \in \mathfrak{A}$ such that every sequence $\{y_j\}_1^\infty$ satisfying one of conditions 1), 2), 3) of Theorem 1 is a basis of E. Then the basis $\{x_j\}_1^\infty$ satisfies, respectively, one of conditions I, II, III of Theorem 1.

3. **Theorem 3.** Let $\{x_j\}_1^\infty$ be a basis of the complete space E, satisfying one of conditions I, II, III of Theorem 1. Then for any index $\beta \in \mathfrak{A}$ there exists an index $\gamma \in \mathfrak{A}$ such that every ω-linearly independent (or complete) sequence $\{y_j\}_1^\infty$ satisfying, respectively, one of the following conditions:

$$1') \quad \sum_{j=1}^{\infty} \frac{|x_j - y_j|_\alpha}{|x_j|_\beta} < \infty \quad (\alpha \in \mathfrak{A});$$

$$2') \quad \text{the set } \left\{ \frac{x_j - y_j}{|x_j|_\beta} \right\}_1^\infty \text{ is bounded and } \frac{x_j - y_j}{|x_j|_\beta} \in V_\gamma \text{ for } j \ge j_0;$$

$$3') \quad \sum_{j=1}^{\infty} \frac{|x_j - y_j|^2_\alpha}{|x_j|^2_\beta} < \infty \quad (\alpha \in \mathfrak{A}),$$

is a basis of E, equivalent to $\{x_j\}_1^\infty$.

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* The space E is called **complete** if every fundamental sequence converges in it.

For Theorem 3 there is a remark analogous to Remark 1. Let us note that Theorems 1 and 3, in the case of uniformly continuous bases, are certain generalizations of known results of M. G. Krein, D. P. Milman, and M. A. Rutman (^{2})* (see also (^{3}), pp. 628-629).

We shall give one more criterion for the stability of an absolute basis, which in some cases (for example, in a Montel space) is more convenient than the criteria given above.

Theorem 4. *Let* $\{x_j\}_1^\infty$ *be an absolute basis in a bornological space* E. *If* $\{y_j\}_1^\infty$ *is an* ω*-linearly independent (or complete) sequence and for some* $\beta \in \mathfrak{A}$ *the set*

$$\left\{ \frac{x_j - y_j}{|x_j|_\beta} \right\}_1^\infty$$

is precompact, then $\{y_j\}_1^\infty$ *forms a basis of* E*, equivalent to* $\{x_j\}_1^\infty$.

Remark 2. Theorems 1-4 carry over to the case when there is no continuous norm in the space E. In this case, if $R(\beta) = \{j : \ |x_j|_\beta = 0\}$, then conditions 1)-3), 1′)-3′) should be extended only to the indices $j \notin R(\beta)$, while for $j \in R(\beta)$ one should put $y_j = x_j$.

Let us note that the general criterion of V. D. Milman for the symmetric stability of minimal and basic sequences (see (^{6})) also admits a certain generalization to a locally convex space**.

Let us also note that Theorems 1 and 3 carry over to the case of defective bases (see (^{5})).

4. Here we give some applications of the results obtained to the problem of quasipower bases in the space A_R of functions analytic in the disk $|z| < R$. In so doing, generalizations of certain results of K. M. Fishman and G. M. Sas' ko (^{8}) are obtained.

Consider the system of functions $h_k(z) = z^k + g_k(z)$ $(k = 0, 1, 2, \ldots)$, where

$$g_k(z) = \sum_{n=0}^{\infty} a_{nk} z^n \quad (k = 0, 1, 2, \ldots).$$

From the results of §§ 2 and 3 of the present note the following theorems follow directly.

Theorem 5. *Let the system of functions* $\{g_k(z)\}_0^\infty$ $(g_k(z) \in A_R;\ k = 0, 1, 2, \ldots)$ *satisfy the following condition: there exists a number* ρ $(0 < \rho < R)$ *such that for every* r $(\rho < r < R)$ *there is a constant* $C(r)$ *for which the inequality*

$$\sup_n |a_{nk}|\, r^n \le C(r) \rho^k \quad (k = 0, 1, 2, \ldots)$$

holds.

If, in addition, one of the following conditions holds:

a)

$$\sup_k \sum_{n=0}^{\infty} |a_{nk}| \rho^{n-k} < 1;$$

b) *the system* $\{h_k(z)\}_0^\infty$ *is* ω*-linearly independent (or complete) in* A_R*, then the system* $\{h_k(z)\}_0^\infty$ *forms a quasipower basis in every* A_r $(\rho < r \leq R)$*.*

Theorem 6. *Let the system of functions* $\{g_k(z)\}_0^\infty$ $(g_k(z) \in A_R,\ k = 0, 1, ...)$*, for which*

$$q(r) = \sup_k \sum_{n=0}^{\infty} |a_{nk}|\, r^{n-k} < \infty \quad (R_1 < r < R),$$

satisfy one of the following conditions: a′) $q(r) < 1$ $(R_1 < r < R)$*;*

b′) the system $\{h_k(z)\}_0^\infty$ is ω-linearly independent (or complete) in at least one $\bar{A}_r$ and

$$\lim_{N\to\infty} \sup_k \sum_{n=N}^{\infty} |a_{nk}|\, r^{n-k} < 1 \qquad (R_1 < r < R).$$

Then the system $\{h_k(z)\}_0^\infty$ forms a quasipower basis in each A_r $(R_1 < r \leq R)$.

We indicate some consequences of the theorems presented above.

Corollary 1. Let

$$h_k(z) = z^k + f^{(k)}(z) \qquad (k = 0, 1, 2, ...),$$

where

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n,$$

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* In a somewhat different direction, the results of M. G. Krein, D. P. Milman, and M. A. Rutman are generalized to the case of countably normed spaces in the paper (4).

** As has recently become known to us, certain results on the stability of bases in locally convex spaces had previously been established by V. D. Milman (an indication of this is given in (7)).

and suppose that for some $\rho > 0$ there exists a constant C such that

$$|a_n| \leq C\rho^n \qquad (n = 0, 1, 2, \ldots).$$

If, in addition, the system $\{h_k(z)\}_0^\infty$ is ω-linearly independent (or complete) in A_r for some r, or if $C \leq qe^{-\rho^2}$ $(q < 1)$, then the system $\{h_k(z)\}_0^\infty$ forms a quasipower basis in each A_r $(r > \rho)$.

Corollary 2. Let $f(z) = \sum_{n=0}^\infty a_n z^n$ $(\in A_R)$, and let $\{\xi_k\}_0^\infty$ be a numerical sequence for which one of the following conditions is satisfied:

1)

$$|\xi_k| \sum_{n=0}^\infty |a_n| r^n \leq q(r) k! \quad (q(r) < 1,\ 0 < r < R,\ k = 0, 1, 2, \ldots);$$

2) $a_0 \neq -k!/\xi_k$, and condition 1) is fulfilled for each r, beginning with some index $k = k(r)$.

Then the system

$$h_k(z) = z^k + \xi_k \int_0^z \int_0^z \cdots \int_0^z f(z)\,(dz)^k \qquad (k = 0, 1, 2, \ldots)$$

forms a quasipower basis in each A_r $(0 < r \leq R)$.

Corollary 3. Let $f(z) = \sum_{n=0}^\infty a_n z^n$ $(\in A_R)$, and let $\{\xi_k\}_0^\infty$ and $\{\lambda_k\}_0^\infty$ be numerical sequences for which

$$m = \max_k |\xi_k| < R; \qquad |\lambda_k| \leq C\rho^k \qquad (k = 0, 1, 2, \ldots)$$

for some ρ $(0 < \rho < R/m)$.

If the system $h_k(z) = z^k + \lambda_k f(\xi_k z)$ $(k = 0, 1, 2, \ldots)$ is ω-linearly independent (or complete) in $A_{R/m}$, or if

$$C \sum_{n=0}^\infty |a_n| (m\rho)^n < 1,$$

then it forms a quasipower basis in each A_r $(\rho < r < R/m)$.

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Note: Figure translations are in progress. See original paper for figures.