AN APPROXIMATION THEOREM FOR ANALYTIC FUNCTIONS AND ITS APPLICATIONS

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 517.537

MATHEMATICS

S. L. KRUSHKAL

AN APPROXIMATION THEOREM FOR ANALYTIC FUNCTIONS AND ITS APPLICATIONS

(Presented by Academician M. A. Lavrentiev on 18 VI 1968)

In this note an approximation theorem for analytic functions is proved and its applications in the theory of quasiconformal mappings are indicated. Its proof is based on certain considerations used in (^{1,3}).

1. For a nonnegative integer q, denote by $A_q(U)$ the complex Banach space of functions $\varphi(z)$ analytic in the disk $U: \ |z| < 1$, with norm

$$\|\varphi\|_{A_q(U)} = \iint_U (1 - |z|^2)^q |\varphi(z)| \, dx \, dy < \infty.$$

Let n be a fixed natural number.

Theorem 1. For any function $\varphi(z) \in A_q(U)$ there exists a sequence of rational functions $r_j(z)$, having, possibly, simple poles, and only at points of the circumference $\gamma: \ |z| = 1$, satisfying the condition $\operatorname{Im}[z^n r_j(z)] = 0$ on γ (when $r_j(z) \neq \infty$) and such that

$$\lim_{j \to \infty} \iint_U (1 - |z|^2)^q |r_j(z) - \varphi(z)| \, dx \, dy = 0. \tag{1}$$

Proof. Consider the space $A_q(U)$ as a real Banach space with the same norm (allowing multiplication of functions $\varphi(z) \in A_q(U)$ only by real numbers) and denote this real space by $\widetilde{A}_q(U)$.

Let α be the set of all rational functions $r(z)$ with simple poles lying on γ, satisfying the condition $\operatorname{Im}[z^n r(z)] = 0$ for $|z| = 1$ (if $r(z) \neq \infty$). The set α is linear in $\widetilde{A}_q(U)$. To prove the theorem it suffices to show that if l is any linear (real) functional in $\widetilde{A}_q(U)$ such that $l(\varphi) = 0$ for all $\varphi \in \alpha$, then $l = 0$ in $\widetilde{A}_q(U)$.

For the given functional $l(\varphi)$ in $\widetilde{A}_q(U)$, put $L(\varphi) = l(\varphi) - il(i\varphi)$. Then $L(\varphi)$ is a complex linear (bounded) functional in $A_q(U)$, and $l(\varphi) = \operatorname{Re} L(\varphi)$. By the

Hahn-Banach and F. Riesz-Steinhaus theorems, the functional $L(\varphi)$ has the form

$$L(\varphi) = \iint_U (1-|z|^2)^q \nu(z)\varphi(z)\,dx\,dy = \iint_U \mu(z)\varphi(z)\,dx\,dy, \tag{2}$$

where $\mu(z)$ is some complex-valued function measurable in U such that $(1-|z|^2)^{-q}\mu(z) = \nu(z) \in L_\infty(U)$. Put

$$h(z) = -\frac{z^{2n-1}-1}{\pi} \iint_{|\zeta|<1} \frac{\mu(\zeta)\,d\xi\,d\eta}{(\zeta - z)(\zeta^{2n-1}-1)}, \qquad h_1(z) = (1+z)h(z), \tag{3}$$

where $\zeta = \xi + i\eta$. Let $a_k = e^{2k\pi i/(2n-1)}$, $k = 0, 1, \dots, 2n-2$. On the basis of the known properties of the integral

$$-\frac{1}{\pi} \iint_U \frac{\mu(\zeta)\,d\xi\,d\eta}{\zeta - z}$$

for $\mu \in L_\infty(U)$ (see (4), Ch. I, § 6), we conclude that the function $h(z)$ has the following properties:

1) $h(a_k) = 0$, $k = 0, 1, \dots, 2n-2$;

2) $h(z)$ is continuous in the entire z-plane, and for every disk $U_R : |z| < R$, $R > 0$, we have

$$|h(z_1) - h(z_2)| \le C(R)|z_1 - z_2|\,|\ln|z_1 - z_2||, \qquad |z_1|, |z_2| \le R, \ C(R) = \text{const}; \tag{4}$$

3) $h(z)$ has generalized derivatives in the sense of S. L. Sobolev, $h_z, h_{\bar z} \in L_p(U_R)$, $p > 1$, with $h_{\bar z} = \mu$ for $z \in U$;

4) $h(z)$ is analytic for $|z| > 1$, and $h(z) = O(|z|^{2n-2})$ near $z = \infty$.

Let $l(\varphi) = 0$ for all $\varphi \in \alpha$. For any fixed $z \in \gamma$, $z \ne a_k$ $(k = 0, 1, \dots, 2n-2)$, the function $\Phi_{(z)}(\xi) = (1+z)(\xi - z)^{-1}(\xi^{2n-1} - 1)^{-1}$ belongs to the set α, and

$$-\pi h_1(z) = L(\Phi_{(z)})\,(z^{2n-1} - 1).$$

Put $h_2(z) = -\pi h_1(z)/(z^{2n-1} - 1)$. The function $h_2(z)$ at the points $a_0, a_1, \dots, a_{2n-2}$ may tend to infinity logarithmically; it is continuous at the remaining points of the circle $|z| = 1$ and bounded at infinity. Since $l(\Phi_{(z)}) = 0$, we have $\operatorname{Re} h_2(z) = 0$ for $|z| = 1$, $z \ne a_k$, $k = 0, 1, \dots, 2n-2$. Hence, by the Poisson formula for harmonic functions, we obtain that $\operatorname{Re} h_2(z) = 0$ everywhere for $|z| \ge 1$. Consequently, for $|z| \ge 1$ we have $h_2(z) = iC$, where C is a real constant. But $h_2(-1) = 0$, and therefore $h_2(z) = 0$ for $|z| \ge 1$. It follows that

$$h_1(z) = h(z) = 0 \qquad \text{for } |z| \ge 1. \tag{5}$$

Denote by $\delta(z) = 1 - |z|$ the distance from the point $z \in U$ to γ, assuming $\delta \le \delta_0 \le e^{-2}$. From (4) and (5) it follows that

$$|h(z)| \le C_1 \delta(z) \ln \frac{1}{\delta(z)},$$

where $C_1 = C(1)$, $|z| < 1$.

We shall prove that for $q \ge 1$ the stronger estimate

$$|h(z)| \le 2C_1 [\delta(z)]^{q+1} \ln \frac{1}{\delta(z)}, \qquad |z| < 1 \tag{6}$$

holds.

For this purpose, for an arbitrary fixed θ, $0 \le \theta \le 2\pi$, consider the function

$$\tilde{h}(z) = -\frac{(1 - ze^{i\theta})^q (z^{2n-1} - 1)}{\pi} \iint \frac{(1 - |\zeta|^2)^q \, \nu(\zeta) \, d\xi \, d\eta}{(1 - \zeta e^{i\theta})^q (\zeta - z)(\zeta^{2n-1} - 1)}. \tag{7}$$

Since the generalized derivative $h_{\bar{z}} - \tilde{h}_{\bar{z}} = 0$ in any disk $|z| \le R$, $R > 0$, we have $h(z) = \tilde{h}(z) + \Phi(z)$, where $\Phi(z)$ is a polynomial of degree $q + 2n - 2$. By virtue of (5), $\tilde{h}(z) = -\Phi(z)$ for $|z| \ge 1$, and from (7) it is clear that $\tilde{h}(z)$ has zeros $z = e^{-i\theta}$ and $z = a_k$, $k = 0, 1, \dots, 2n-2$, of total multiplicity $q + 2n - 1$. Consequently, $\Phi(z) \equiv 0$, i.e. $h(z) \equiv \tilde{h}(z)$. Hence, applying to the integral on the right-hand side of (7) an inequality analogous to (4), and putting $\theta = -\arg z$, we obtain (6).

Following L. Ahlfors (^{1}), we now consider an infinitely differentiable function $\Lambda(t)$ such that $\Lambda(t) = 0$ for $0 \le t \le 1$, $\Lambda(t) = 1$ for $t \ge 2$, and for an arbitrary positive $\varepsilon > 0$ put

$$\lambda_\varepsilon(z) = \Lambda[\varepsilon^{-1} (\ln \ln 1/\delta(z))^{-1}]. \tag{8}$$

Taking into account that $|\partial \delta(z)/\partial \bar{z}| \le 1$, $\Lambda'(t) = 0$ outside the interval $1 \le t \le 2$, and $(\ln \ln 1/\delta)^{-1} = t \le 2\varepsilon$ for $t \in [\varepsilon, 2\varepsilon]$, we shall have

$$|\partial \lambda_\varepsilon / \partial \bar{z}| \le C_2 \varepsilon^{-1} \delta^{-1} (\ln 1/\delta)^{-1} (\ln \ln 1/\delta)^{-2} \le 4C_2 \varepsilon \, \delta^{-1} (\ln 1/\delta)^{-1}, \tag{9}$$

where $C_2 = \max |\Lambda'(t)|$ for $1 \le t \le 2$. From (6) and (9) we have

$$|\lambda_{\varepsilon \bar{z}} h(z)| \le 8C_1 C_2 \varepsilon (1 - |z|^2)^q, \qquad |z| < 1. \tag{10}$$

By Green's formula, in view of the finiteness of $\lambda_\varepsilon(z)$, for any function $\varphi \in A_q(U)$ we obtain

$$\iint_U \lambda_\varepsilon \mu \varphi \, dx \, dy = \iint_U \lambda_\varepsilon (h\varphi)_{\bar{z}} \, dx \, dy - \iint_U \lambda_{\varepsilon \bar{z}} \, h\varphi \, dx \, dy$$

and, consequently, by virtue of (10),

$$\left|\iint_U \lambda_\varepsilon(z)\mu(z)\varphi(z)\,dx\,dy\right| \le C_3\varepsilon\|\varphi\|_{A_q(U)}, \qquad C_3 = 8C_1C_2. \tag{11}$$

From (11), in view of the equality $\lim \lambda_\varepsilon(z) = 1$ in the limit as $\varepsilon \to 0$, taking (2) into account, we obtain that $L(\varphi) = 0$, whence it follows that also $l(\varphi) = 0$. Thus the theorem is proved.

Remark. As is seen from the proof, the assertion of Theorem 1 remains valid also under the condition that the functions $r_j(z)$ may have poles only at the points of some set $E \subset \gamma$, everywhere dense on γ and containing the points $a_0, a_1, \dots, a_{2n-2}$.

2. We indicate some applications of Theorem 1. Since for $|z| = 1$ we have $z = e^{i\psi}$ and $\operatorname{Im}[r(z)\,dz^2] = -\operatorname{Im}[z^2r(z)]\,d\psi^2$, it follows from Theorem 1 that

Theorem 2. *For any function $\varphi(z) \in A_q(U)$ there exists a sequence of rational functions $r_j(z)$, which may have simple poles, and moreover only at points of the circle $\gamma : |z| = 1$, satisfying the condition $\operatorname{Im}[r_j(z)\,dz^2] = 0$ on γ, and such that*

$$\lim_{j\to\infty} \|r_j(z) - \varphi(z)\|_{A_q(U)} = 0.$$

Theorem 2 makes it possible to establish some properties of extremal quasiconformal mappings.

For fixed q, denote by Ω_q the set of all homeomorphisms $w = \omega(z)$ of the circle γ onto $\gamma' \colon |w| = 1$ that are boundary values of quasiconformal homeomorphisms $w = f^\mu(z)$ of the disk U onto $U' \colon |w| < 1$ with complex characteristics $\mu(z) = k\varphi(z)/|\varphi(z)|$, where $\varphi \in A_q(U)$, and $k = \|\varphi\|_{A_q(U)} < 1$. Without loss of generality, we shall assume that $\omega(e^{2\pi li/3}) = e^{2\pi li/3}$, $l = 0, 1, 2$. Denote by $w = f^{\mu_m}(z)$ the mapping (with characteristic $\mu_m(z)$) minimizing the dilatation $K(f)$ in the class of all quasiconformal homeomorphisms of the disk U onto U', sending the given (distinct) points $z_1, z_2, \dots, z_m$ of the circle γ to the points $\omega(z_1), \omega(z_2), \dots, \omega(z_m)$, respectively, for a given $\omega \in \Omega_q$, and call such mappings extremal.* Then the characteristic $\mu_m(z) = k_m r_m(z)/|r_m(z)|$, where $k_m = \text{const} < 1$, and $r_m(z)$ is a rational function which may have simple poles at the points $z_1, z_2, \dots, z_m$ and is such that $\operatorname{Im}[r_m(z)\,dz^2] = 0$ for $|z| = 1$, while $r_m(z)$ is determined up to a constant positive factor (see ([2],[5],[7])). We choose this factor so that $\|r_m\|_{A_q(U)} = k_m$. Conversely, by virtue of what was proved in ([2],[7]), every mapping $w = f^{\mu_m}(z)$ with characteristic $\mu_m(z) = \|r_m\|_{A_q(U)} r_m(z)/|r_m(z)|$ of the indicated form is extremal in the class of quasiconformal homeomorphisms of the disk U onto U' sending the points $z_1, z_2, \dots, z_m$ of the circle γ, which are poles of $r_m(z)$, to the points $f^{\mu_m}(z_1), f^{\mu_m}(z_2), \dots, f^{\mu_m}(z_m)$, respectively.

From Theorem 1 the assertion follows.

Every mapping $w = f^{\mu}(z)$ with characteristic $\mu = \bar{k}\varphi/|\varphi|$, $k < 1$, with $\varphi \in A_q(U)$, is the limit, in the sense of uniform convergence in $\overline{U}$, of some sequence of extremal mappings $w = f^{\mu_m}(z)$ with characteristics $\mu_m(z) = k_m\overline{r_m(z)}/|r_m(z)|$ of the above-indicated form, where $r_m \to \varphi$ in $A_q(U)$.

Here the question remains open whether, for $q \geqslant 1$, every mapping $w = f^{\mu}(z)$ with characteristic $\mu = k\varphi/|\varphi|$, $\varphi \in A_q(U)$, is extremal (i.e., minimizes the dilatation $K(f)$) in the class of all quasiconformal mappings f of the disk U onto U' that coincide with f^{μ} on γ. (This has been proved only for $q = 0$ (6).)

Institute of Mathematics
Siberian Branch
Academy of Sciences of the USSR

Received
9 VI 1968

References Cited

[1] L. Ahlfors, *Am. J. Math.*, **86**, No. 2, 413 (1964).
[2] L. Ahlfors, L. Bers, *Spaces of Riemann Surfaces and Quasiconformal Mappings*, IL, 1961, pp. 9 and 104.
[3] L. Bers, *J. Anal. Math.*, **14**, 1 (1965).
[4] I. N. Vekua, *Generalized Analytic Functions*, Moscow, 1959.
[5] S. L. Krushkal' , *Siberian Mathematical Journal*, **8**, No. 2, 313 (1967).
[6] K. Strebel, *Comm. Math. Helv.*, **39**, Nos. 2-3, 81 (1964).
[7] O. Teichmüller, *Abh. Preuss. Akad. Wiss., Math.-Naturwiss. Kl.*, **22**, 1 (1940).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

---

* The dilatation of a mapping f with characteristic $\nu(z) = f_{\bar{z}}/f_z$, where $\|\nu\|_{L_\infty(U)} = k(f) < 1$, is the quantity