ON MAXIMUM PRINCIPLES FOR EQUATIONS OF MIXED ELLIPTIC-HYPERBOLIC TYPE OF THE SECOND KIND

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Abstract

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MATHEMATICS

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ON MAXIMUM PRINCIPLES FOR EQUATIONS OF MIXED ELLIPTIC-HYPERBOLIC TYPE OF THE SECOND KIND

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Consider the equation

$$K_m[u] \equiv u_{xx} + \operatorname{sgn} y \cdot |y|^m u_{yy} + \frac{M(x;y)}{|y|^\beta} u_x + \frac{N(x;y)}{|y|^\alpha} u_y + \frac{F(x;y)}{|y|^\gamma} u = 0 \quad (K_m)$$

in a domain D, bounded by: 1) a simple Jordan arc σ, situated in the upper half-plane and resting on the axis $y = 0$ at the points $O(0;0)$, $A(a,0)$, $a > 0$; 2) the characteristics OC and AC. Let D_1 and D_2 be respectively the subdomains of ellipticity and hyperbolicity of the equation (K_m), OA its line of transition; $0 < m < 2$; $M, N, F \in C^{(0)}(\overline{D})$; $M, N \in C^{(1)}(\overline{D}_2)$; $F \leq 0$ in D_1. Particular cases of (K_m) are the equations studied in a mixed domain

$$u_{xx} + \operatorname{sgn} y \cdot |y|^m u_{yy} = 0, \qquad 0 < m < 1 \quad (^1); \quad (\Sigma_m)$$

$$u_{xx} + y u_{yy} + a u_y = 0, \qquad a < 1 \quad (^{2-5}); \quad (S_1^\alpha)$$

$$u_{xx} + \operatorname{sgn} y \cdot |y|^m u_{yy} + a|y|^{m-1} u_y = 0, \qquad 0 < m < 2, \quad m/2 < \alpha < 1 \quad (^6). \quad (S_m^\alpha)$$

We shall say that for equation (K_m) in the domain D (in the domain D_2), in a certain class $[K]$ of its solutions, maximum principle I (maximum principle II) holds in the following case: let $u \in [K]$ and $u|_{OC} \equiv 0$; then $\max_{\overline{D}} |u|$ $(\max |u|)$ is attained on $\sigma \cup AC$ (on the characteristic AC).

§ 1. Consider the equation

$$E[u] \equiv u_{\xi\eta} + a(\xi;\eta) u_\xi + b(\xi;\eta) u_\eta + c(\xi;\eta) u = 0 \quad (E)$$

in the characteristic triangle $O_0C_0A_0$ (the domain Δ, $O_0(0;0)$, $C_0(0;-\xi_0)$, $A_0(\xi_0;-\xi_0)$, $\xi_0>0$); $a, a_\xi, b, c \in C^{(0)}(\Delta \cup O_0C_0)$.

We shall say that in the domain Δ the coefficients of equation (E) satisfy conditions (L), if: 1) $a>0$ on O_0C_0; 2)

$$a(P) > \int_{PQ} \{2|\gamma_2| + \beta|c_2|\}\, d\xi$$

in Δ, and conditions (M), if in $\Delta \cup O_0C_0$: 1) $a>Q$; 2)

$$a(Q)\beta(Q) > \int_{PQ} |\gamma|\, d\xi.$$

Here PQ is a segment of an arbitrary characteristic $\eta = \text{const}$,

$$P \in O_0C_0, \quad Q \in \Delta; \quad \beta = \exp\left\{\int_0^\xi b\, d\xi\right\}; \quad h = a_\xi + ab - c = h_1 + h_2, \quad h_1 \geq 0 \text{ in } \Delta;$$

$$\gamma = \gamma_1 + \gamma_2, \quad \gamma_1 = -\beta h_1, \quad \gamma_2 = -\beta h_2; \quad c = c_1 + c_2, \quad c_1 \geq 0.$$

We shall call the function $u(\xi;\eta)$ a solution of equation (E) of class $[W_0]$, if: $E[u] \equiv 0$ in Δ, $u \in C^{(2)}(\Delta)$, $u \in C^{(0)}(\overline{\Delta})$, $u \in C^{(1)}(\Delta \cup O_0C_0)$.

Lemma 1 (the characteristic principle of the absolute extremum for hyperbolic equations). Let $u(\xi;\eta)$ be a solution of equation (E) of class $[W_0]$, $u \neq 0$ in Δ, and $u|_{O_0C_0} = 0$. Then $\max_\Delta |u|$ cannot be attained in Δ, but is attained on the characteristic $\overline{A_0C_0}$.

1. Suppose the contrary. Let $\max_\Delta |u|$ be attained at some point $Q \in \Delta$. We write equation (E) in the form

$$(\beta u_\eta)_\xi + (\alpha\beta u)_\xi + \gamma u = 0$$

and integrate this equation over the segment PQ ($P \in O_0C_0$, $\eta_P = \eta_Q$). We obtain

$$\beta(Q)u_\eta(Q) = \left[-a(P)u(Q) - \int_{PQ} [u(Q)-u]\gamma_2\, d\xi - u(Q)\int_{PQ} \beta C_2\, d\xi\right] +$$

$$+ \int_{PQ} [u(Q)-u]\gamma_1\, d\xi - u(Q)\int_{PQ} \beta C_1\, d\xi; \tag{1}$$

$$\beta(Q)u_\eta(Q) = -\beta(Q)a(Q)u(Q) - \int_{PQ} \gamma u\, d\xi. \tag{2}$$

By virtue of conditions (L) and (M), respectively, from (1) and (2) it follows that

$$u(Q)u_\eta(Q) < 0, \tag{3}$$

which is impossible.

2. Suppose that $\max_\Delta |u|$ is not attained on $\overline{A_0C_0}$. Then, by the preceding part, it is attained only on the open segment O_0A_0 at some point Q. Consequently, in Δ, in some neighborhood of the point Q, there exists a point Q' at which $u \neq 0$ and

$$|u(Q')| > \max_{A_0C_0} |u|. \tag{4}$$

Draw through the point Q a segment $O_0'A_0' \parallel O_0A_0$ ($O_0' \in O_0C_0$, $A_0' \in C_0A_0$), and denote by Δ' the open triangular domain $O_0'C_0A_0'$. $\max_{\Delta'} |u|$ cannot be attained at a point belonging to $\Delta' \cup O_0'A_0'$, since, by the preceding part, at such a point one would have $u \cdot u_\eta < 0$. Hence $\max_{\Delta'} |u|$ is attained only on $\overline{A_0'C_0}$, and

$$|u(Q')| \leq \max_{A_0'C_0} |u|,$$

which contradicts (4) and completes the proof of the lemma.

It can be shown that $\max_\Delta |u|$ is attained only on the characteristic A_0C_0, if

$$|u(A)| < \max_{A_0C_0} |u|.$$

Denote by $(E_m^{\lambda\mu})$ equation (E) in which

$$a = -\sigma/2(\xi+\eta) + f(\xi;\eta)/|\xi+\eta|^{\sigma+\lambda}, \quad b = -\sigma/2(\xi+\eta) + g(\xi;\eta)/|\xi+\eta|^{\sigma+\lambda},$$

$$c = r(\xi;\eta)/|\xi+\eta|^{\mu}, \quad \sigma = m/(2-m), \quad 0 < m < 1, \quad \lambda < 2(1-m)/(2-m),$$
$$\mu < (4-3m)/2(2-m); \quad f,\ (\xi+\eta)f_\xi,\ g,\ r \in C^{(0)}(\overline{\Delta}).$$

Lemma 2. For equation $(E_m^{\lambda\mu})$, the assertion of Lemma 1 is valid if the length of the segment O_0A_0 is sufficiently small.

The proof follows from the fact that conditions (L) are satisfied if the length of the segment O_0A_0 is sufficiently small.

We note that, under the change of variables

$$\xi = x - \frac{2}{2-m}(-y)^{(2-m)/2}, \quad \eta = -x - \frac{2}{2-m}(-y)^{(2-m)/2},$$

equation (K_m) is transformed into an equation of the form (E), and the triangle OAC into a triangle of type Δ.

§ 2. We shall call a function $u(x;y)$ a solution of equation (K_m) of class $[W]$ if:

$$K_m[u] \equiv 0 \quad \text{in } D_1 \cup D_2, \quad u \in C^{(2)}(D_1 \cup D_2), \quad u \in C^{(0)}(\overline{D}),$$

$$u \in C^{(1)}(\overline{D} \setminus \overline{OA}).$$

Theorem 1. For equation (K_m) in the domain D (in the domain D_2), in the class $[W]$ of its solutions there holds the maximum principle I (the maximum principle–

II), if the length of the transition line is sufficiently small and $0 < m < 1$, $\beta < 1 - m/2$, $\alpha < 1 - m$, $\gamma < 1 - 3m/4$.

Proof. Let $u(x;y)$ be an arbitrary solution of equation (K_m) of class $[W]$, $u|_{\overline{OC}} \equiv 0$ and $u \not\equiv 0$ in D. Then, by the sufficient smallness of the transition line, by Lemma 2, $\max_{\overline{D}2}|u|$ is attained on $\overline{AC}$, and maximum principle II is proved.

Let $\max_{\overline{D}}|u|$ be attained in $\overline{D}_2$. Consequently, it is attained on $\overline{AC}$.

Let $\max_{\overline{D}}|u|$ be attained in $\overline{D}_1$. Then, by the known property of elliptic equations, it is attained on $\bar{\sigma} \cup OA$, and, if on OA, then, by the preceding, also on AC. Thus, $\max_{\overline{D}}|u|$ is attained on $\bar{\sigma} \cup AC$, and the theorem is completely proved.

Theorem 2. *For equation (K_m) in the domain D (in the domain D_2), in the class $[W]$ of its solutions, maximum principle I (maximum principle (II)) is valid if its coefficients are such that the coefficients of the corresponding equation of the form (E) satisfy in Δ the conditions (I) or (M).*

The proof is analogous to Theorem 1 and follows from Lemma 1.

We note that in Theorem 2 the requirement of continuity on the characteristic AC of the coefficients of equation (K_m) and of the first partial derivatives of the functions $M(x;y)$ and $N(x;y)$ is superfluous.

Corollary of Theorem 2. For an arbitrary length of the transition line, for equation (Σ_m) with $0 < m < 2$ and for equation (S_m^α) with $0 < m < 2$, $\alpha < m/2$, in the class $[W]$ maximum principles I and II are valid.

§ 3. Suppose a one-to-one correspondence has been established between all points of the characteristic $\overline{AC}$ and some set of points $E \subset \sigma$; Q_1 and Q_2 are arbitrary corresponding points $(Q_1 \in E,\ Q_2 \in \overline{AC})$, $E_1 \equiv \bar{\sigma} \setminus E$, and at each point belonging to the set E, the arc σ has a normal.

Problem Φ. Find a solution $u(x;y)$ of equation (K_m) of class $[W]$ from the data:

1. $u|_{E_1} = \varphi_1$. 2. $\partial u/\partial n|_E = \varphi_2$. 3. $u|_{\overline{OC}} = \psi$. 4. $u(Q_1) - u(Q_2) = g(Q_1)$.

Here $\varphi_1, \varphi_2, \psi, g$ are functions continuous on the indicated parts of the boundary of the domain D; n is the interior normal. Boundary condition 4 defines a "curvilinear jump of compaction" (^{7,8}).

Theorem 3. *For equation (K_m), under the conditions of Theorems 1 and 2, the solution of problem Φ is unique.*

Suppose the contrary. Let $u(x;y)$ be a solution of problem Φ for equation (K_m), satisfying homogeneous boundary conditions, but $u \not\equiv 0$ in D. By homogeneous boundary condition 3, $\max_{\overline{D}} |u|$ is attained on $\bar{\sigma} \cup AC$. Taking into account homogeneous boundary conditions 1 and 2, we obtain that $\max_{\overline{D}} |u|$ cannot be attained on $\bar{\sigma}$ and, consequently, is attained on $\overline{AC}$. But then, by homogeneous boundary condition 4, it is attained on $\bar{\sigma}$, which, as has already been shown, is impossible. The contradiction obtained proves the theorem.

Corollary of Theorem 3. For equations (Σ_m) and (S_m^α), under the conditions of the corollary to Theorem 2, the solution of problem Φ is unique.

We note that for equation (S_1^α), when $\alpha < 0$, there exists and is unique a solution of the Dirichlet problem in the class of functions continuous in $\overline{D}$ and having a continuous derivative with respect to y on the transition line (^{4}).

Remark 1. We have considered maximum principles I and II with data on the characteristic $\overset{\frown}{OA}$. These principles are formulated analogously with data on the characteristic AC. For their proof, in equation (E) one should interchange the roles of the variables ξ and η. Maximum principles I and II with data on the characteristic AC hold for equation (K_m) under the conditions of Theorem 1 and for equations (Σ_m) and (S_m^α) under the conditions of Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.