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Abstract

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MATHEMATICS

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ASYMPTOTICS AS $t \to \infty$ OF SOME SOLUTIONS OF A SYSTEM OF TWO QUASILINEAR EQUATIONS

(Presented by Academician M. V. Keldysh on 15 VII 1968)

1. Let there be given a quasilinear hyperbolic system satisfying the Lax condition (see below)

$$\partial u/\partial t + \partial f(u,v)/\partial x = 0; \qquad \partial v/\partial t + \partial \varphi(u,v)/\partial x = 0 \tag{1}$$

and such values $u_-, v_-;\ u_+, v_+$ that there exists a continuous solution $u = \hat{u}(x/t),\ v = \hat{v}(x/t)$ of the problem of the decay of a discontinuity, i.e. of the Cauchy problem

$$u(x,0) = \begin{cases} u_+, & x > 0, \\ u_-, & x < 0; \end{cases} \qquad v(x,0) = \begin{cases} v_+, & x > 0, \\ v_-, & x < 0. \end{cases} \tag{2}$$

Let now $u_0(x)$ and $v_0(x)$ be continuously differentiable functions "smoothing" the initial data (2), i.e. satisfying the condition

$$\mathbf{A}_1. \quad \lim_{x\to\pm\infty} u_0(x) = u_\pm; \qquad \lim_{x\to\pm\infty} v_0(x) = v_\pm.$$

Then, as $\varepsilon \to 0$, the functions $u_0(x/\varepsilon)$ and $v_0(x/\varepsilon)$ tend to the initial data (2), and it is natural to expect that the solution $u_\varepsilon(x,t), v_\varepsilon(x,t)$ of the system (1) with initial conditions

$$u_\varepsilon(x,0) = u_0(x/\varepsilon); \qquad v_\varepsilon(x,0) = v_0(x/\varepsilon) \tag{3}$$

should, as $\varepsilon \to 0$, tend respectively to $\hat{u}(x/t)$ and $\hat{v}(x/t)$. We shall prove the validity of this assertion under the following additional condition on the functions $u_0(x), v_0(x)$:

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$\mathbf{A}_2$.

The vector $u_0'(x), v_0'(x)$, for every x, does not vanish, is not an eigenvector of the matrix

$$\begin{pmatrix} f_u & f_v \\ \varphi_u & \varphi_v \end{pmatrix}$$

at the point $u_0(x), v_0(x)$, and tends to zero as $|x| \to \infty$.

Theorem 1. *If the functions $u_0(x), v_0(x)$ satisfy conditions $\mathbf{A}_1$ and $\mathbf{A}_2$, then for all $t > 0$, x, $\varepsilon > 0$ there exist functions $u_\varepsilon(x,t), v_\varepsilon(x,t)$ satisfying (1), assuming the initial values (3), and tending as $\varepsilon \to 0$ to $\hat{u}(x/t)$ and $\hat{v}(x/t)$.*

In view of the invariance of system (1) with respect to similarity transformations, this theorem is equivalent to the following assertion:

Theorem 2. *If the functions $u_0(x)$ and $v_0(x)$ satisfy conditions $\mathbf{A}_1$ and $\mathbf{A}_2$, then for all $t > 0$, x there exist functions $u(x,t), v(x,t)$ satisfying the system (1) and assuming for $t = 0$ the values*

$$u(x,0) = u_0(x); \qquad v(x,0) = v_0(x), \tag{4}$$

for any x_0, ξ there exist the limits

$$\lim_{t\to\infty} u(x_0 + t\xi, t) = \hat{u}(\xi); \qquad \lim_{t\to\infty} v(x_0 + t\xi, t) = \hat{v}(\xi),$$

where $\hat{u}(x/t)$ and $\hat{v}(x/t)$ are the solution of the problem on the decay of a discontinuity (2).

2. Let us outline the proof of Theorem 2. To this end we rewrite system (1) in the Riemann invariants $s(u,v)$ and $\sigma(u,v)$ ([1], [2])

$$\partial s/\partial t + \lambda(s,\sigma)\partial s/\partial x = 0; \qquad \partial\sigma/\partial t + \mu(s,\sigma)\partial\sigma/\partial x = 0 \tag{5}$$

(where $\lambda > \mu$). Then Lax' s condition means that $\partial\lambda/\partial s \neq 0$, $\partial\mu/\partial\sigma \neq 0$, and one may assume that

$$\partial\lambda/\partial s > 0; \qquad \partial\mu/\partial\sigma > 0. \tag{6}$$

The Cauchy data (2) pass into the values

$$s(x,0) = \begin{cases} s_+, & x > 0, \\ s_-, & x < 0; \end{cases} \qquad \sigma(x,0) = \begin{cases} \sigma_+, & x > 0, \\ \sigma_-, & x < 0; \end{cases} \tag{7}$$

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where $s_+ > s_-$ and $\sigma_+ > \sigma_-$ (otherwise the Cauchy problem (7) will not have a continuous solution $\hat{s}(x/t), \hat{\sigma}(x/t)$), and the Cauchy data (4) become functions

$$s(x,0) = s_0(x); \qquad \sigma(x,0) = \sigma_0(x), \tag{8}$$

where, by virtue of A_1 and A_2:

B_1. $\lim\limits_{x\to\pm\infty} s_0(x) = s_\pm; \quad \lim\limits_{x\to\pm\infty} \sigma_0(x) = \sigma_\pm$.

B_2. $\partial s_0/\partial x > 0; \quad \partial\sigma_0/\partial x > 0$, and $(\partial s_0/\partial x)^2 + (\partial\sigma_0/\partial x)^2$ tends to zero as $|x| \to \infty$.

We now consider the system

$$\partial x/\partial\sigma = \lambda(s,\sigma)\partial t/\partial\sigma, \tag{9}$$

$$\partial x/\partial s = \mu(s,\sigma)\partial t/\partial s,$$

obtained by formal inversion of system (5), and the initial data for this system, prescribed on the curve P: $s = s_0(\eta)$; $\sigma = \sigma_0(\eta)$, where the functions $s_0(\eta)$ and $\sigma_0(\eta)$ have the form (8)

$$t|_P = 0; \qquad x|_P = \eta. \tag{10}$$

Theorem 2 follows from the following assertions:

C_1. The solution $t(s,\sigma), x(s,\sigma)$ of system (9) and of the Cauchy problem (10) exists and is bounded at every point inside the triangle ABC (see Fig. 1), bounded by the curve P and by the segments AB and BC, parallel to the axes σ and s.

C_2. Inside ABC, $\partial t/\partial\sigma > 0$; $\partial t/\partial s < 0$, and $|\partial t/\partial\sigma|$, $|\partial t/\partial s|$ tend uniformly to infinity as one approaches the segments AB and BC.

C_3. t tends to infinity as one approaches the segments AB and BC, uniformly outside any neighborhood of the points A and C.

C_4. The Jacobian

$$J = \frac{\partial t}{\partial s}\frac{\partial x}{\partial\sigma} - \frac{\partial t}{\partial\sigma}\frac{\partial x}{\partial s}$$

does not vanish in the triangle ABC, and therefore, after inverting the functions $x(s,\sigma)$, $t(s,\sigma)$, we obtain a single-valued and continuously differentiable solution $s(x,t), \sigma(x,t)$ satisfying the initial data (8).

C_5. The functions $s(x_0 + \xi t, t)$ and $\sigma(x_0 + \xi t, t)$ as $t \to \infty$ tend to:

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Fig. 1 and Fig. 2

Figure 1: Fig. 1 and Fig. 2

$$(s_-, \sigma_-), \qquad \text{if } \mu(s_-, \sigma_-) > \xi; \tag{11}$$

$$(s_-, \sigma_0), \qquad \text{where } \mu(s_-, \sigma_0) = \xi, \text{ if } \mu(s_-, \sigma_-) < \xi < \mu(s_-, \sigma_+); \tag{12}$$

$$(s_-, \sigma_+), \qquad \text{if } \mu(s_-, \sigma_+) < \xi < \lambda(s_-, \sigma_+); \tag{13}$$

$$(s_0, \sigma_+), \qquad \text{where } \lambda(s_0, \sigma_+) = \xi, \text{ if } \lambda(s_-, \sigma_+) < \xi < \lambda(s_+, \sigma_+); \tag{14}$$

$$(s_+, \sigma_+), \qquad \text{if } \lambda(s_+, \sigma_+) < \xi, \tag{15}$$

i.e., to the solution $\hat{s}(x/t), \hat{\sigma}(x/t)$ of the problem of decay of the discontinuity (7) (see ([1], [2])).

3. Let us outline the proof of assertions C_1–C_4. At any interior point D of the triangle ABC, the solution $x(s, \sigma)$ and $t(s, \sigma)$ of the system (9) and of the Cauchy problem (10) depends only on the Cauchy data on the arc EH (where ED and DH are characteristics of equation (9), i.e., straight lines parallel to the σ- and s-axes). And since on this arc x and t are bounded together with their derivatives with respect to s and σ,

Fig. 1 **Fig. 2**

these quantities are also bounded at the point D, whence C_1 follows.

Let us prove that at D

$$\partial t/\partial \sigma > 0; \qquad \partial t/\partial s < 0. \tag{16}$$

These inequalities hold on the arc EH (which follows from B_2), and therefore hold in some neighborhood of EH. Replacing, if necessary, the point D by some point D' inside EDH, we may assume that the inequalities (16) hold inside EDH. Let us prove that they also hold at the point D. Eliminating x from equations (9), we obtain:

$$(\lambda - \mu) \frac{\partial^2 t}{\partial s\, \partial \sigma} + \frac{\partial \lambda}{\partial s} \frac{\partial t}{\partial \sigma} - \frac{\partial \mu}{\partial \sigma} \frac{\partial t}{\partial s} = 0, \tag{17}$$

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and, putting $y = \partial t/\partial\sigma$, we obtain for y the ordinary differential equation on the segment HD

$$(\lambda - \mu)\frac{dy}{ds} + \frac{\partial\lambda}{\partial s} y = \frac{\partial\mu}{\partial\sigma}\frac{\partial t}{\partial s}. \tag{18}$$

Here the right-hand side is nonpositive (since $\partial\mu/\partial\sigma > 0$, while $\partial t/\partial s$, negative inside EDH, is nonpositive on DH). Therefore, integrating (18) from the point H to D, it is easy to obtain the estimate

$$y(D) = \partial t(D)/\partial\sigma > \Phi_0(D)\,\partial t(H)/\partial\sigma > 0, \tag{19}$$

where $\Phi_0(D)$ depends on the values of $\lambda - \mu$ and $\partial\lambda/\partial s$ on the interval HD and is uniformly bounded below in the triangle ABC.

Analogously to (18), one obtains the equation for $z = \partial t/\partial s$:

$$(\lambda - \mu)\frac{dz}{d\sigma} - z\frac{\partial\mu}{\partial\sigma} = -\frac{\partial\lambda}{\partial s}\frac{\partial t}{\partial\sigma}, \tag{20}$$

integrating which along the segment ED and taking into account that the right-hand side in (20) is also nonpositive, we obtain

$$z(D) = \partial t(D)/\partial s < \Phi_1(D)\,\partial t(E)/\partial s < 0. \tag{21}$$

If we now use (19) and (21) to estimate the right-hand sides in (20) and (18) and again estimate $y(D)$ and $z(D)$, then, since as $E \to A$ and $H \to C$ the quantities $|\partial t/\partial s|$, $|\partial t/\partial\sigma|$ at the points E and H tend to infinity, we obtain the proof of C_2, and therefore also of C_3.

Assertion C_4 follows at once from inequality (16) and equation (9):

$$J = \frac{\partial t}{\partial s}\frac{\partial x}{\partial\sigma} - \frac{\partial x}{\partial s}\frac{\partial t}{\partial\sigma} = (\lambda - \mu)\frac{\partial t}{\partial s}\frac{\partial t}{\partial\sigma} \neq 0.$$

4. It suffices to prove C_5 for the case $\xi = 0$, to which one easily passes from arbitrary ξ by the change of independent variables $x' = x - t\xi,\ t' = t$.

We shall restrict ourselves to proving formula (12); expressions (11) and (13)–(15) are proved similarly. This means that we consider the case when on the segment AB μ changes sign. It is necessary to prove that, for fixed x and $t \to \infty$, the trajectories $s(x, t)$, $\sigma(x, t)$ tend to the point E_0 on the segment AB at which $\mu = 0$.

Denote by L and K the level lines $\lambda = 0$ and $\mu = 0$, respectively; evidently, K passes through the point E_0, while L may be absent (see Fig. 2). Since in the triangle ABC

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$$\frac{\partial s}{\partial x} = \frac{1}{(\lambda - \mu)\, \partial t/\partial \sigma} > 0; \qquad \frac{\partial \sigma}{\partial x} = \frac{-1}{(\lambda - \mu)\, \partial t/\partial s} > 0, \tag{22}$$

it follows from equations (5) that the relative disposition of the directions of the trajectories $s(x,t)$, $\sigma(x,t)$ and of the axes s, σ must be, in the various regions into which the curves L and K divide ABC, as shown in Fig. 2. From C_1 it follows that as $t \to \infty$ the point $s(x,t), \sigma(x,t)$ must tend to the broken line ABC. Therefore each trajectory $s(x,t)$, $\sigma(x,t)$ tends, as $t \to \infty$, to some limit $\bar{s}(x), \bar{\sigma}(x)$. And since from C_2 and (22) it follows that $\partial s/\partial x$ and $\partial \sigma/\partial x$ tend uniformly to zero as $t \to \infty$, this limit must not depend on x and must coincide with the point E_0, which proves C_5.

It is clear that if one returns from the Riemann invariants $s(u,v)$, $\sigma(u,v)$ to the variables u, v, then C_5 becomes Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.

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