Theory of the Figure of Hydrostatic-Equilibrium Rotating Planets

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Theory of the Figure of Hydrostatic-Equilibrium Rotating Planets

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The liquid rotating planets Jupiter and Saturn are probably in a state close to hydrostatic equilibrium. From Bernoulli' s equation and the equation of state of matter it follows that, in hydrostatic equilibrium, equipotential surfaces are at the same time surfaces of constant pressure p and constant density ρ. In particular, one of them is the surface of the planet, $p = \text{const} = 0$. Therefore the figure of such a planet is entirely determined by the form of the equipotential surfaces.

Taking into account symmetry with respect to the planet' s axis of rotation, let us write the external potential of the force of gravity in the form ([1])

$$U_e(r,t)\frac{GM_1}{r} = \left\{1 - \sum_{n=1}^{\infty}\left(\frac{a_1}{r}\right)^{2n} J_{2n}(1)P_{2n}(t) + \frac{q}{3}\left(\frac{r}{a_1}\right)^3 [1 - P_2(t)]\right\}, \quad (1)$$

where G is the gravitational constant; M_1 and a_1 are the mass and equatorial radius of the planet; $q = \omega^2 a_1^3/GM_1$; ω is the angular velocity of rotation; r and $t = \cos\vartheta$ are the coordinates of the point under consideration, lying outside the planet; $P_{2n}(t)$ are Legendre polynomials. The last term in (1) represents the centrifugal potential. The relative multipole moments $J_n(1)$ are determined by the relation

$$J_n(1) = \frac{-1}{Ma^n}\int \rho(r',t')(r')^n P_n(t')\,d\tau, \quad (2)$$

here $\rho(r,t)$ is the density distribution in the planet, and the integration is carried out over the volume of the planet where $\rho \neq 0$. The index 1 indicates that the corresponding quantity is considered at the surface of the planet.

Let b_1 and e_1 denote the polar radius and the flattening of the planet, $e_1 = (a_1 - b_1)/a_1$. The flattening e_1 proves to be proportional to the quantity q

and is a small parameter of the problem, while the moments have the order of magnitude $J_{2n}(1) \sim e_1^n$.

In the zero approximation the planet is a sphere; in the first approximation, an ellipsoid of revolution (Clairaut spheroid) ([2]); in the second, a deformed ellipsoid of revolution (Darwin-de Sitter-Jeffreys spheroid) ([3,4]).

The model of the planet must be constructed so as to satisfy the prescribed values of the multipole moments and radius. For Saturn, observations give $J_4(1)$ with an accuracy of $\pm 6.7\%$, and the flattening is $e_1 = 0.0978$. Since the moment $J_4(1)$ is a quantity of second order, the terms of the third approximation contribute to $J_4(1)$ on the order of 10% and must be taken into account.

The main results of the theory of the spheroid of the third approximation reduce to the following.

The equation of the equipotential surfaces outside and inside the planet is sought in the form

$$\begin{aligned} r(a,t) = a[1 - e\cos^2\vartheta - ({}^3/_8 e^2 + k)\sin^2 2\vartheta \\ + {}^1/_4({}^1/_2 e^3 + h)(1 - 5\sin^2\vartheta)\sin^2 2\vartheta]. \end{aligned} \tag{3}$$

The parameters e, k, h characterize the shape of the equipotential surfaces and are functions of the equatorial radius a of this surface. The radius a is a parameter of the family of equipotential surfaces. In particular, for $a = a_1$ formula (3) gives the equation of the outer surface, i.e., the figure of the planet. If the terms e^3 and h are neglected, then spheroid (3) becomes the Darwin-de Sitter-Jeffreys spheroid ([3,4]).

It can be shown that the parameters of the spheroid on the outer surface are related to the moments $J_n(1)$ by the relations

$$\begin{aligned} J_2(1) &= {}^2/_3 e_1 - {}^1/_3 q + {}^3/_7 e_1 q + {}^8/_{21} k_1 - {}^{50}/_{294} e_1^2 q + {}^{40}/_{147} e_1 k_1 + {}^8/_{21} q k_1 - {}^2/_{21} h_1, \\ J_4(1) &= -{}^4/_5 e_1^2 + {}^4/_7 e_1 q - {}^{32}/_{35} k_1 + {}^4/_5 e_1^3 - {}^{50}/_{49} e_1^2 q + {}^{3616}/_{2695} e_1 k_1 \\ &\quad + {}^{208}/_{385} q k_1 - {}^{192}/_{385} h_1, \\ J_6(1) &= {}^8/_7 e_1^3 - {}^{20}/_{21} e_1^2 q + {}^{128}/_{77} e_1 k_1 - {}^{160}/_{231} q k_1 + {}^{80}/_{231} h_1. \end{aligned} \tag{4}$$

Instead of the independent variable a, it is convenient to introduce s—the mean radius (the radius of a sphere whose volume is equal to the volume enclosed within the equipotential surface under consideration), i.e., to pass to a new parameter s of the family of surfaces.

$$s = a\,(1 - {}^1/_3 e + {}^1/_9 e^2 - {}^5/_{81} e^3 - {}^8/_{15} k - {}^{26}/_{105} h + {}^{32}/_{315} e k)\,. \tag{5}$$

Substituting (5) into (3) and introducing Legendre polynomials instead of trigonometric functions, we obtain

$$
\begin{aligned}
r(s,t) = s[1 &- {}^4/_{45}e^2 - {}^{52}/_{567}e^3 - {}^{32}/_{315}ek \\
&- ({}^2/_3 e + {}^{23}/_{63}e^2 + {}^4/_{27}e^3 + {}^8/_{21}k - {}^2/_{21}h + {}^{152}/_{315}ek)P_2 \\
&+ ({}^{12}/_{35}e^2 + {}^{32}/_{35}k + {}^4/_{11}e^3 + {}^{192}/_{385}h + {}^{32}/_{105}ek)P_4 \\
&- ({}^{40}/_{231}e^3 + {}^{80}/_{231}h)P_6].
\end{aligned} \tag{6}
$$

The gravitational potential is the integral $\int \rho(\mathbf{r}')\,d\tau'/|\mathbf{r}-\mathbf{r}'|$, taken over the volume of the entire planet. Let $\mathbf{r}$ be the radius vector of an arbitrary internal point. The equipotential surface passing through this point divides the planet into an inner region and an outer shell. In these regions the integrand may be expanded in series of Legendre polynomials, respectively in descending $(n<0)$ and ascending $(n>0)$ powers of the modulus of the radius vector r^n. After integration, to terms of third order inclusive, we obtain

$$
\begin{aligned}
\frac{3}{4\pi G\rho_0}U(r,t) = \frac{s^3}{r}D - {}^2/_5\frac{s^5}{r^3}SP_2 + {}^{12}/_{35}\frac{s^7}{r^5}PP_4 - {}^8/_{21}\frac{s^9 H}{r^7}P_6 \\
+ {}^3/_2 s^2 E - {}^2/_5 r^2 TP_2 + {}^{32}/_{105}\frac{r^4}{s^2}QP_4 - {}^{80}/_{1001}\frac{r^6}{s^4}IP_6 + {}^1/_3 m(1-P_2)r^2.
\end{aligned} \tag{7}
$$

Here s is the mean radius of the equipotential surface passing through the point (r,t) under consideration; ρ_0 is the mean density of the planet; $\delta(s)=\rho(s)/\rho_0$ is the relative density; $m = 3\omega^2/4\pi G\rho_0$. The quantities D, S, P, H, E, T, Q, I are relative multipole moments of the inner and outer parts of the planet. Introducing the relative mean radius $\beta = s/s_1$, we have

$$
D(\beta) = \beta^{-3}\int_0^\beta \delta(z)\,d[z^3], \qquad d[z^3] = \frac{d[z^3]}{dz}\,dz,
$$

$$
S(\beta) = \beta^{-5}\int_0^\beta \delta(z)\,d\left[z^5\left(e + {}^1/_6 e^2 + {}^4/_7 k + {}^2/_9 e^3 - {}^1/_7 h + {}^4/_3 ek\right)\right],
$$

$$
P(\beta) = \beta^{-7}\int_0^\beta \delta(z)\,d\left[z^7\left(e^2 + {}^8/_9 k + {}^{16}/_{33}h + {}^1/_3 e^3 + {}^{40}/_{297}ek\right)\right],
$$

$$H(\beta) = \beta^{-9} \int_0^\beta \delta(z)\, d\left[z^9\left(e^3 + {}^{192}/_{143} ek + {}^{30}/_{143} h\right)\right],$$
$$E(\beta) = \beta^{-2} \int_\beta^1 \delta(z)\, d\left[z^2 - {}^4/_{45} z^2\left(e^2 + {}^{61}/_{63} e^3 + {}^8/_7 ek\right)\right],$$
$$T(\beta) = \int_\beta^1 \delta(z)\, d\left(e + {}^9/_{14} e^2 + {}^4/_7 k - {}^1/_7 h + {}^8/_{21} e^3 + {}^4/_7 ek\right), \qquad (8)$$
$$Q(\beta) = \beta^2 \int_\beta^1 \delta(z)\, d\left[z^{-2}\left(k + {}^6/_{11} h + {}^{14}/_{33} ek\right)\right],$$
$$I(\beta) = \beta^4 \int_\beta^1 \delta(z)\, d\left[z^{-4}(h - 4ek)\right].$$

On an equipotential surface $U(r,t) = \text{const} = U_0$. Substitute (6) into (7). Since the resulting expression is equal to U_0 for any $t = \cos\vartheta$, the coefficients of $P_n(t)$ must be equal to zero for $n \neq 0$. As a result we obtain a system of 4 integro-differential equations, the first of which relates the mean radius s of the equipotential surface to the value of the potential $U_0(s)$ on it, while the next three equations determine the parameters e, k, h as functions of s.

$$\begin{aligned} 3U_0/s^2 4\pi G\rho_0 = {} & \left(1 + {}^8/_{45} e^2 + {}^{64}/_{315} ek + {}^{584}/_{2835} e^3\right) D \\ & - {}^4/_{25}\left(e + {}^{13}/_{14} e^2 + {}^4/_7 k\right) S + {}^3/_2 E + {}^8/_{75}\left(e + {}^{19}/_{42} e^2 + {}^4/_7 k\right) T \\ & + {}^1/_3 m\left(1 + {}^4/_{15} e + {}^2/_{63} e^2 + {}^{16}/_{105} k\right); \end{aligned} \qquad (9)$$

$$\begin{aligned} & \left(e + {}^{31}/_{42} e^2 + {}^4/_7 k - {}^1/_7 h + {}^{38}/_{63} e^3 + {}^{44}/_{105} ek\right) D \\ & \quad - {}^3/_5\left(1 + {}^4/_7 e + {}^{10}/_7 e^2 - {}^{16}/_{35} k\right) S + e\, {}^{24}/_{49} P - {}^3/_5\left(1 - {}^8/_{21} e + {}^{32}/_{105} k\right) T \\ & \quad - {}^{256}/_{735} eQ - {}^1/_2 m\left(1 + {}^{20}/_{21} e + {}^{38}/_{63} e^2 + {}^{16}/_{15} k\right) = 0; \end{aligned} \qquad (10)$$

$$\begin{aligned} & \left(3e^2 - 8k - {}^{48}/_{11} h + {}^{277}/_{77} e^3 + {}^{2152}/_{231} ek\right) D \\ & \quad - 6\left(e + {}^{159}/_{154} e^2 + {}^{52}/_{55} k\right) S + 3\left(1 + {}^{200}/_{231} e\right) P \\ & \quad - {}^{32}/_5 kT + {}^8/_3\left(1 - {}^{160}/_{231}\right) Q = 0; \end{aligned} \qquad (11)$$

$$D\left(e^3 + {}^{32}/_7 ek - {}^{10}/_7 h\right) - {}^3/_7\left(e^2 + 8k\right) S + {}^{15}/_7 eP + {}^{11}/_7 H + {}^{32}/_{21} eQ + {}^{30}/_{91} I = 0. \qquad (12)$$

Specifying the density distribution along some direction, for example $\rho(r, t_0)$, from the system (9)–(12) one can find the form of the level surfaces, in particular the outer surface, and the entire function $\rho(r,t)$ to third-order accuracy. Then the multipole moments J_n can be calculated. Comparison of the observed and calculated values of J_n imposes a constraint on the possible distributions $\rho(r,t)$ in the planet.

The system (9)–(12), in combination with Bernoulli' s equation, also makes it possible to calculate the figure of a planet consisting of matter with a given equation of state, and to select models satisfying the observed J_n.

As an example, let us consider two limiting cases. For a planet with constant density ($\rho=\rho_0,\ \delta=1$), the system of equations (10)–(12) has the solution

$$e=\bar{e}+{}^{3}/_{14}\bar{e}^2+{}^{37}/_{98}\bar{e}^3,\qquad k=h=0,\qquad \bar{e}={}^{5}/_{4}m=15\omega^2/16\pi G\rho_0,\quad (13)$$

which corresponds to the Maclaurin ellipsoid. The multipole moments according to (2) are

$$J_2={}^{2}/_{5}\bar{e}\left(1-{}^{2}/_{7}\bar{e}+{}^{8}/_{49}\bar{e}^2\right),\qquad J_4=-{}^{12}/_{35}\left(1-{}^{4}/_{7}\bar{e}\right)\bar{e},\qquad J_6={}^{8}/_{21}\bar{e}^3. \quad (14)$$

For a planet whose entire mass is concentrated at the center, i.e., whose density has the form of a delta function, the solution of system (10)–(12) will be

$$e=\tilde{e}+\frac{3}{5}\tilde{e}^3,\qquad k=\frac{3}{8}\tilde{e}^2-\frac{3}{20}\tilde{e}^3,\qquad h=\frac{19}{10}\tilde{e}^3,\qquad \tilde{e}=\frac{1}{2}m\beta^3. \quad (15)$$

In particular, at the surface of such a planet $\beta=1$ and $\tilde{e}=\frac{1}{2}m$. The multipole moments in this model are equal to zero: $J_2=J_4=J_6=\cdots=0$.

The theory set forth can be used to construct more accurate models of Jupiter and Saturn; in particular, it makes it possible for the first time to compute theoretically the moment J_6.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.