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Abstract

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MATHEMATICS

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ON THE PROBLEM OF THE DECAY OF A DISCONTINUITY FOR A SYSTEM OF TWO QUASILINEAR EQUATIONS

(Presented by Academician M. V. Keldysh on 15 VII 1968)

1. In this note an example is constructed satisfying the Lax condition (see below) of a hyperbolic quasilinear system

$$\frac{\partial u}{\partial t} + \frac{\partial f(u,v)}{\partial x} = 0, \qquad \frac{\partial v}{\partial t} + \frac{\partial \varphi(u,v)}{\partial x} = 0, \tag{1}$$

for which the shock adiabat $A(u_1, v_1)$, i.e. the set of points u_2, v_2 satisfying, for some ω, the relation

$$\omega(u_1 - u_2) = f(u_1, v_1) - f(u_2, v_2); \quad \omega(v_1 - v_2) = \\ = \varphi(u_1, v_1) - \varphi(u_2, v_2), \tag{2}$$

is a bounded set in the plane u_2, v_2. Therefore it is easy to specify such values u_-, v_- and u_+, v_+ that system (1) has no self-similar solution of the problem of the decay of a discontinuity, i.e. no solution of the form $u = u(x/t)$, $v = v(x/t)$, satisfying the Cauchy data

$$u(x,0) = \begin{cases} u_+, & x > 0, \\ u_-, & x < 0; \end{cases} \qquad v(x,0) = \begin{cases} v_+, & x > 0, \\ v_-, & x < 0. \end{cases} \tag{3}$$

Theorem. There exists a constant $\eta_0 > 1$ such that, if $u_+ > \eta_0^2 u_-$, then for system (1), where

$$f = 3 \ln u + v, \qquad \varphi = 2/u, \tag{4}$$

the problem of the decay of a discontinuity (3) has no self-similar solution $u = u(x/t)$, $v = v(x/t)$.

The proof of this theorem is based on the following lemma.

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Lemma. There exist constants $\eta_0 > 1$ and C such that, for system (1), (4), the shock adiabat $A(u_1, v_1)$, i.e. the set of points u_2, v_2 satisfying (2), is contained in the rectangle

$$u_1/\eta_0 < u_2 < \eta_0 u_1, \qquad |v_1 - v_2| < C.$$

2. Let us prove the lemma. Eliminating ω from (2), we obtain that $A(u_1, v_1)$ coincides with the zeros of the function

$$K(u_2, v_2) = (u_2 - v_1)(\varphi(u_2, v_2) - \varphi(u_1, v_1)) - \\ -(v_2 - v_1)(f(u_2, v_2) - f(u_1, v_1)) = 0. \quad (5)$$

If we use (4) and then solve (5) with respect to v_2, we obtain the equation

$$v_2 = v_1 + \frac{3}{2}\ln\eta \pm \sqrt{\Delta},$$

where $\eta = u_2/u_1$, $\Delta = \frac{9}{4}\ln^2\eta - 2(\eta-1)^2/\eta$. But $\Delta \geq 0$ only when $1/\eta_0 < \eta < \eta_0$, where $\eta_0 > 1$ is the root of the equation $\Delta = 0$. Therefore (5) has real roots u_2, v_2 only when $u_1/\eta_0 < u_2 < \eta_0 u_1$, and it is easily verified that, for these values of u_2, the difference $|v_1 - v_2|$ is bounded in modulus.

3. Before proving the theorem, let us recall how a self-similar solution of the problem of the decay of a discontinuity is defined. According to ([1,2]), a self-similar solution of the problem of the decay of a discontinuity is understood to mean piecewise continuous functions $u(\xi)$, $v(\xi)$ (where $\xi = x/t$), having piecewise continuous derivatives and satisfying the conditions:

1°. $u(\pm\infty) = u_\pm$; $v(\pm\infty) = v_\pm$.

2°. If at a point ξ the functions $u(\xi)$, $v(\xi)$ are continuous and have nonzero derivatives, then they satisfy the equations

$$\xi u'_\xi = f'_u u'_\xi + f'_v v'_\xi, \\ \xi v'_\xi = \varphi'_u u'_\xi + \varphi'_v v'_\xi. \quad (6)$$

Formula (6) is obtained if one substitutes into (1) the functions $u(x/t)$, $v(x/t)$ and then sets $x/t = \xi$. From (6) it is clear that the tangent $u'(\xi)$, $v'(\xi)$ to the curve $u = u(\xi)$, $v = v(\xi)$, considered in the u, v-plane, is at each point an eigenvector of the matrix

$$\begin{pmatrix} f'_u & f'_v \\ \varphi'_u & \varphi'_v \end{pmatrix}, \quad (7)$$

and ξ is the corresponding eigenvalue. Curves in the u, v-plane touching at each of their points an eigenvector of the matrix (7) at that point are called **characteristics**. The naturally arising Lax condition consists in the requirement

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that along each characteristic the corresponding eigenvalue vary strictly monotonically. If this condition is fulfilled, then on any characteristic $u(\xi)$, $v(\xi)$ are single-valued functions of ξ, and therefore $u(x/t)$, $v(x/t)$ is a single-valued and continuous solution of the system (1).

3°. If at $\xi = \omega$ the functions $u(\xi)$, $v(\xi)$ have a discontinuity, with

$$u(\omega + 0) = u_2; \quad u(\omega - 0) = u_1;$$
$$v(\omega + 0) = v_2; \quad v(\omega - 0) = v_1,$$

then u_1, v_1, u_2, v_2 and ω are related by relation (2), i.e., the point u_2, v_2 lies on the shock adiabat $A(u_1, v_1)$.

Equation (2) is obtained if the system (1) is written in the x, t-plane in the form

$$\operatorname{div}(u, f(u, v)) = 0; \qquad \operatorname{div}(v, \varphi(u, v)) = 0$$

and these expressions are integrated over a neighborhood of the straight line $x/t = \omega$. If the solution $u(x/t)$, $v(x/t)$ has a discontinuity at $x/t = \omega$, then we say that the solution contains at $x/t = \omega$ a shock wave.

4°. Only stable shock waves are allowed.

The stability condition consists in requiring that in the x, t-plane three characteristics of the system (1) impinge on the discontinuity line $x/t = \omega$: either two from the left and one from the right, in which case $\omega < \lambda_1,\ \mu_1;\ \mu_2 < \omega < \lambda_2$, or one from the left and two from the right, in which case $\mu_1 < \omega < \lambda_1;\ \omega > \lambda_2,\ \mu_2$. Here μ_1, λ_1 and μ_2, λ_2 ($\mu_1 < \lambda_1,\ \mu_2 < \lambda_2$) are respectively the eigenvalues of the matrix (7) at the points (u_1, v_1) and (u_2, v_2).

4. Let us outline the proof of the theorem. Simple calculations show that for the system (4) $\lambda = 2/u,\ \mu = 1/u$, and the corresponding characteristics have the form $u = e^{-2(v-s)}$ and $u = e^{-(v-\sigma)}$, where s, σ ($-\infty < s, \sigma < \infty$) are parameters specifying the characteristic. In Fig. 1 the characteristics passing through some point u_1, v_1 are shown; the arrows indicate the directions of increase of the corresponding eigenvalues, and the dashed curve is the shock adiabat $A(u_1, v_1)$.

Let us now consider what conditions u_-, v_- and u_+, v_+ must satisfy in order that the problem of the decay of a discontinuity have a self-similar solution. To do this, let us trace the change of $u(\xi)$ as ξ increases from $-\infty$ to $+\infty$.

As ξ increases on an interval where $u(\xi)$, $v(\xi)$ are continuous and have nonzero derivatives $u'(\xi)$, $v'(\xi)$, the point $u(\xi), v(\xi)$ moves along a certain characteristic of system (4) in the direction of increase of the corresponding eigenvalue (equal, according to (6), to ξ). But since $\lambda = 2/u$, $\mu = 1/u$, under such motion $u(\xi)$ must decrease. It is easy to show, following (1) or (2), that for any system satisfying Lax' s condition, the solution $u(\xi), v(\xi)$ cannot contain more than two shock waves. But, according to the lemma, at each shock wave the limit

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Fig. 1

Figure 1: Fig. 1

on the right from the discontinuity point $x/t = \omega$, $u_2 = u(\omega + 0)$, can exceed the limit on the left $u_1 = u(\omega - 0)$ by no more than a factor of η_0. Therefore, as ξ varies from $-\infty$ to $+\infty$, the function $u(\xi)$ cannot increase by more than a factor of η_0^2, which proves the theorem.

Fig. 1

5. Example (4) is a system defined in the half-plane $u > 0$, $-\infty < v < \infty$. It is easy to construct an example of a system possessing analogous properties but defined on the entire u, v-plane. For this it is enough to set

$$f = 5\left[uv\sqrt{u^2+1} + u^2 + \ln\left(u + v\sqrt{u^2+1}\right)\right] + v, \tag{8}$$

$$\varphi = -{}^{16}/_{3}\left[2u^3 + 3u + 2(u^2+1)\sqrt{u^2+1}\right].$$

The investigation of this system is analogous to the investigation of system (4) carried out above, but differs in more cumbersome calculations.

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Note: Figure translations are in progress. See original paper for figures.

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