ON THE EXPONENTIAL TOPOLOGY

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Abstract

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ON THE EXPONENTIAL TOPOLOGY

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The present paper is devoted to the space of closed sets. We give only definitions and results. In what follows all spaces are assumed to be completely regular, unless explicitly stated otherwise. The terminology is the same as in ([3],[7]). Let X be a topological space. Denote by $F(X)$ the space of all nonempty closed subsets of the space X in the Vietoris topology. A base of the exponential topology is formed by sets of the form $\langle U_1, \dots, U_n \rangle =$

$$= \{L \in F(X) \mid L \subset \bigcup_{i=1}^{n} U_i;\ L \cap U_i \neq \emptyset, \quad i = 1, 2, \dots, n\},$$

where the sets $U_1, U_2, \dots, U_n$ are open in the space X. The following subspaces of the space $F(X)$ are considered: $C(X) = \{L \in F(X) \mid L \text{ is bicompact}\}$; $J_n(X) = \{B \in F(X) \mid \operatorname{card} B \leq n\}$, where n is a fixed natural number; $J(X) = \bigcup_{n=1}^{\infty} J_n(X)$.

In the paper ([4]) V. Ivanova proved that the space $F(N)$, where N is the space of natural numbers in the usual topology, is nonnormal, whence follows

Proposition 1. Let X be a weakly paracompact space. The space $F(X)$ is normal if and only if the space X is bicompact.

When is the space $C(X)$ normal?

Theorem 1. The following properties are equivalent:

a) X is a paracompact p-space;

b) $J_n(X)$ is a paracompact p-space for every natural number n;

c) $C(X)$ is a paracompact p-space.

Theorem 1 is based on the following two assertions:

Proposition 2. A mapping $f : X \to Y$ of the space X onto Y is perfect if and only if the mapping $f_C : C(X) \to C(Y)$, where $f_C\Phi = f\Phi$ for every element $\Phi \in C(X)$, is perfect.

Theorem 2. The mapping $\varphi_n : X^n \to J_n(X)$, where $\varphi_n(x_1, x_2, \dots, x_n) = \{x_1, x_2, \dots, x_n\}$, is open-closed, continuous, and finite-to-one.

Theorem 3. The space X is complete in the sense of Čech if and only if the space $C(X)$ is complete in the sense of Čech.

Theorem 4. Let the space X have a point-countable base. Then the space $C(X)$ also has a point-countable base.

Theorem 5. For every T_1-space X, the weight of the space $F(X)$ is equal to the weight of the Wallman extension ωX of the space X.

Let $A \subset X$. We say that the external π-weight of the set A relative to the space X does not exceed the cardinal number τ, if there exists a family $\Omega = \{U_\alpha \mid \alpha \in \theta,\ U_\alpha \cap A \neq \emptyset,\ \operatorname{cord} \theta \leq \tau\}$ of sets open in X such that for every open set U in X, where $U \cap A \neq \emptyset$, there exists $U_\alpha \in \Omega$ for which $U_\alpha \subseteq U$. In this case $\pi - p(A, X) \leq \tau$. If $A = X$, then $\pi - p(X, X) = \pi - p(X)$.

Proposition 3. For every topological space X we have

$$\pi - p(X) = \pi - p(C(X)) = \pi - p(F(X)).$$

Let $A \subset X$. Put $\chi_X(A)$ for the character of the set A in the space X.

Lemma 1. Let $A \in F(X)$; $\chi_{F(X)}(A) \leq \tau$ if and only if $\chi_X(A) \leq \tau$ and $\pi - p(A, X) \leq \tau$.

Lemma 1 underlies the proof of the following two theorems.

Theorem 6. For any bicompactum X the following conditions are equivalent:

1) for every element $A \in F(X)$ we have $\chi_{F(X)}(A) \leq \tau$;
2) for every set $B \subset X$ there exists an everywhere dense in $\overline{B}$ set B' of cardinality $\leq \tau$ and $\chi_X([B]) \leq \tau$.

Theorem 7. The space $C(X)$ satisfies the first axiom of countability if and only if:

a) the space X is of countable type*;
b) every bicompact subset $\Phi \subset X$ is perfectly normal and separable.

Corollary 1. For any bicompactum X the following conditions are equivalent:

1) the space $F(X)$ satisfies the first axiom of countability;
2) the space X is perfectly normal and hereditarily separable.

Corollary 2. A bicompactum X is metrizable if and only if the space $F(X \times X)$ satisfies the first axiom of countability.

Let X be an arbitrary topological space. Denote by $I(X)$ the totality of all isolated points of the space X. Put $S(X) = X \setminus I(X)$. Our aim is to indicate for which spaces X the space $F(X)$ satisfies the first axiom of countability.

Theorem 8. The space $F(X)$, where X is normal, satisfies the first axiom of countability if and only if the following conditions hold:

a) the space X is perfectly normal;

b) the set $I(X)$ is countable;

c) the set $S(X)$ is countably compact, hereditarily separable, and $\chi_X(S(X)) \leq \aleph_0$.

Theorem 9. The following conditions are equivalent:

a) the space X is of countable type;

b) the space $C(X)$ is of point-countable type**;

c) the space $C(X)$ is of countable type.

Proposition 4. Let $\chi_{F(X)}(A) \leq \tau$ for any element $A \in F(X)$. Then the cardinality of the space $F(X)$ does not exceed the cardinal number 2^τ.

Theorem 10. For any paracompact p-space X the following conditions are equivalent:

a) the space X is metrizable;

b) the space $C(X)$ is metrizable;

c) the space $C(X)$ is perfectly normal;

d) the space $J(X)$ is perfectly normal;

e) the space $J_n(X)$ is perfectly normal for some natural number $n \geq 2$.

A mapping $\theta : X \to F(Y)$ is lower (upper) semicontinuous if for every open (closed) set $A \subset Y$ the set

$$\{x \in X \mid \theta x \cap A \neq \emptyset\}$$

is open (closed) in X.

Some results of E. Michael from ([8],[9]) admit the following generalization.

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* The space X is called a space of countable type if for an arbitrary bicompactum $F \subset X$ there is a bicompactum $\Phi \subset X$ such that $F \subset \Phi$ and $\chi_X(\Phi) \leq \aleph_0$ (see ([2],[6])).

** The space X is called a space of point-countable type if there exists a cover of this space by bicompacta of countable character (see ([2],[6])).

Theorem A. For every T_1-space X the following conditions are equivalent:

1) X is normal and τ-paracompact*;

2) for every lower semicontinuous mapping $\theta : X \to F(Y)$, where Y is an arbitrary complete metric space of weight $\leq \tau$, there exist mappings $\varphi : X \to C(Y)$ and $\psi : X \to C(Y)$ such that:

a) $\varphi x \subset \psi x \subset \theta x$ for every point $x \in X$;

b) the mapping φ is lower semicontinuous;

c) the mapping ψ is upper semicontinuous;

3) for every lower semicontinuous mapping $\theta : X \to F(Y)$, where Y is an arbitrary complete metric space of weight τ, there exists an upper semicontinuous mapping $\psi : X \to C(Y)$, where $\psi x \subset \theta x$ for every point $x \in X$;

4) into every open covering of the space X of cardinality τ one can inscribe some closed conservative covering.

Theorem B. For every T_1-space X the following conditions are equivalent:

a) X is normal, τ-paracompact, and $\dim X \leq m$;

b) for every lower semicontinuous mapping $\theta : X \to F(Y)$, where Y is an arbitrary complete metric space of weight τ, there exists an upper semicontinuous mapping $\psi : X \to C(Y)$ such that $\psi x \subset \theta x$ and ψx consists of no more than $m + 1$ points for every point $x \in X$.

Denote by H a separable Hilbert space. Then the following holds.

Theorem B′. For every T_1-space X the following conditions are equivalent:

a) X is normal and $\dim X \leq m$;

b) for every lower semicontinuous mapping $\theta : X \to C(H)$ there exists an upper semicontinuous mapping $\psi : X \to C(H)$ such that $\psi x \subset \theta x$ and ψx consists of no more than $m + 1$ points for every point $x \in X$.

Theorem C. Let $f : X \to Y$ be an open mapping of a metric space X onto a metric space Y with complete (in the metric specified on X) inverse images of points. Then for every continuous mapping $g : Z \to Y$ of a zero-dimensional metric space Z onto Y there exists a mapping $h : Z \to X$ such that $g = f \circ h$.

Corollary C. Let $f : X \to Y$ be an open mapping of a metric space X onto a paracompactum Y with complete (in the metric specified on X) inverse images of points. If $\dim Y \leq m$, then there exists a subspace $X_1 \subseteq X$ such that $fX_1 = Y$ and the mapping $f_1 = f|X_1$ is perfect and $m + 1$-fold.

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* The space X is τ-paracompact, where $\tau \geq \aleph_0$, if into every open covering of cardinality τ one can inscribe some locally finite open covering.

Note: Figure translations are in progress. See original paper for figures.