STABILITY OF PULSE SYSTEMS WITH FREQUENCY MODULATION

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Abstract

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MATHEMATICS

A. Kh. GELIG

STABILITY OF PULSE SYSTEMS WITH FREQUENCY MODULATION

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1. Consider a nonlinear pulse system with one pulse element (p.e.), producing instantaneous pulses whose frequency is modulated by the signal $\sigma(t)$ ([1,2]). The mathematical description of such a system can be represented in the form of the nonlinear integral equation

$$\sigma(t) = f(t) - \int_{-0}^{t} \gamma(t-\lambda)\varphi(\lambda)\,d\lambda. \tag{1}$$

Here $\gamma(t)$ is the impulse transition function of the reduced continuous linear part (c.l.p.), the function $f(t)$ describes the natural oscillations of the c.l.p., and $\varphi(t)$ is the signal at the output of the p.e., which can be represented in the form

$$\varphi(t) = \sum_{k=0}^{\infty} \delta_k(t); \qquad \delta_k(t) = \begin{cases} \delta(t-t_k)\operatorname{sign}\sigma_k, & \text{if } |\sigma_k| > \Delta, \\ 0, & \text{if } |\sigma_k| \le \Delta \end{cases} \tag{2}$$

($\sigma_k \equiv \sigma(t_k - 0)$; Δ is a positive constant characterizing the magnitude of the dead zone of the p.e.; $\delta(t)$ is the

The time instants t_k are determined by the formula

$$t_{k+1} = t_k + T(|\sigma_k|), \qquad k = 0, 1, 2, \dots, \qquad t_0 = 0, \tag{3}$$

in which $T(x)$ is a prescribed continuous positive monotonically decreasing function on $[0, +\infty)$, describing the law of frequency-pulse modulation, and

$$\lim_{x \to +\infty} T(x) = T_\infty > 0. \tag{4}$$

It is assumed that the functions $f(t)$ and $\gamma(t)$ are given for $t \ge 0$, are absolutely continuous, and satisfy the conditions

$$|\gamma(t)| \le \gamma_0 \exp(-\varepsilon_0 t), \qquad |d\gamma/dt| \le \gamma_1 \exp(-\varepsilon_0 t) \tag{5}$$

($\gamma_0, \gamma_1, \varepsilon_0$ are positive constants),

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} df/dt = 0, \qquad f(t) \in L_j[0+\infty). \tag{6}$$

We shall assume that the transfer function of the c.l.p.

$$\chi(p) = \int_0^\infty \gamma(t) \exp(-pt)\, dt$$

satisfies the condition

$$\lim_{p\to\infty} p\chi(p) = 0.$$

(The case when this equality is not fulfilled was studied in ([2]).)

Let r_0 be the largest root of the equation

$$r_0 = \gamma_0/(1 - \exp(-\varepsilon_0 T(r_0))). \tag{7}$$

Introduce the notation $r_1 = r_0\gamma_1/\gamma_0$, $T_0 = T(r_0)$.

Theorem 1. *If there exist constants $\varepsilon > 0$ and $\nu \geqslant 0$ such that for all real ω the inequality*

$$\nu + \mathrm{Re}\big[(\varepsilon + i\omega)\chi(i\omega)\big] \geqslant 0 \quad (i = \sqrt{-1}) \tag{8}$$

is satisfied, and the estimate

$$\frac{\varepsilon\nu}{2(1 - e^{-\varepsilon T_0})^2} + \frac{\varepsilon r_0}{e^{\varepsilon T_0} - 1} + (1 - e^{-\varepsilon(\Delta + r_0)/r_1})r_1 < \varepsilon\Delta, \tag{9}$$

holds, then as $t \to +\infty$ the solution $\sigma(t)$ of equation (1) tends to zero, whatever the function $f(t)$ satisfying conditions (6) may be.

2. Let us now consider the critical case of a single zero root. Then instead of equation (1) we have the equation

$$\sigma(t) = f(t) + f_0 - \int_{-0}^{t} [\gamma(t - \lambda) + \rho]\varphi(\lambda)\, d\lambda, \tag{10}$$

in which $f_0 = \text{const}$, $\rho = -\gamma(0) = \text{const} > 0$, and the functions $f(t)$, $\gamma(t)$, and $\varphi(t)$ are the same as in equation (1).

Theorem 2. *Suppose there exist constants $\varepsilon > 0$ and $\nu \geqslant 0$ such that for all real ω the inequality*

$$\rho + \nu + \mathrm{Re}\big[(\varepsilon + i\omega)\chi(i\omega)\big] \geqslant 0$$

is satisfied, and estimate (9) holds, in which $T_0 = T(r_0)$, $r_1 = \gamma_1(r_0 - \rho)/2\gamma_0$, and r_0 is the largest root of the equation $r_0 = \rho + 2\gamma_0/(1 - \exp(-\varepsilon_0 T(r_0)))$. Then, as $t \to +\infty$, the solution $\sigma(t)$ of equation (10) has a limit belonging to the interval $[-\Delta, \Delta]$, whatever the constant f_0 and the function $f(t)$ satisfying conditions (6) may be.

Theorems 1 and 2 answer the question posed in (4) concerning the extension of V. M. Popov' s general frequency criterion to systems with frequency-pulse modulation.

3. Proof of Theorem 1 is carried out according to the scheme developed in (5), supplemented by the following two new points: a) the a priori radius of the attraction sphere of the solution $\sigma(t)$ is estimated recurrently, which makes it possible, in contrast to (5), to take into account the form of the function $T(x)$; b) between the output of the L.E. and the input of the P.N.L.Ch. a unit block with transfer function

$$\frac{p + \varepsilon}{p + \varepsilon}$$

is introduced, which makes it possible to obtain the frequency criterion (8), more general than that in (5).

Let us briefly present the course of the argument. From equations (1), (2), by virtue of conditions (4)–(6), it is easy to obtain the estimate

$$\overline{\lim}_{t\to\infty}|\sigma(t)| \leqslant \tau_1 = \gamma_0/(1 - \exp(-\varepsilon_0 T_\infty)).$$

Therefore there exists a number $N_1 > 0$ such that $|\sigma(t)| < \tau_1 + 1$ for $t > N_1$, and then from equations (1), (2) we obtain the estimate

$$\overline{\lim}_{t\to\infty}|\sigma(t)| \leqslant \tau_2 = \gamma_0/(1 - \exp(-\varepsilon_0 T(\tau_1 + 1))).$$

Therefore there exists $N_2 > N_1$ such that $|\sigma(t)| < \tau_2 + \frac{1}{2}$ for $t > N_2$. Continuing the reasoning in this way, we obtain sequences $\{N_n\}$, $\{\tau_n\}$ such that

$$\tau_{n+1} = \gamma_0/(1 - \exp(-\varepsilon_0 T(\tau_n + 1/n)))$$

and $|\sigma(t)| < \tau_n + 1/n$ for $t > N_n$. From the monotonicity of the function $T(x)$ it follows that $\tau_n \leqslant \tau_{n-1}$. Since $\tau_n > 0$, there exists $\lim_{t\to\infty} \tau_n = \tau$, which satisfies the equation

$$\tau = \gamma_0/(1 - \exp(-\varepsilon_0 T(r))).$$

Thus,

$$\overline{\lim}_{t\to\infty} |\sigma(t)| \leqslant r_0.$$

Differentiating–

differentiating equation (1) for $t \neq t_k$ $(k = 0, 1, 2, ...)$ (here and below the letter t with subscripts denotes the instants of time determined by formula (3)), it is easy to obtain the estimate $\overline{\lim}_{\substack{t\to\infty \\ t\neq t_k \ (k=0,1,2,...)}} |d\sigma/dt| \leq r_1$. Therefore, for any $\delta > 0$ there exists such a $t_q > 0$ that, for $t > t_q$,

$$|\sigma(t)| \leq r_0 + \delta, \qquad |d\sigma/dt| \leq r_1 + \delta \quad (t \neq t_k). \tag{11}$$

For $t > t_q$, equation (1) can be written in the form

$$\sigma(t) = g(t) - \int_{t_q-0}^{t} \gamma_\varepsilon(t-\lambda)\eta(\lambda)\, d\lambda,$$

where

$$g(t) = f(t) - \int_{-0}^{t_q-0} \gamma(t-\lambda)\varphi(\lambda)\, d\lambda, \qquad \gamma_\varepsilon(t) = \gamma(t) + \varepsilon\, d\gamma/dt, \tag{12}$$

and the function $\eta(t)$ satisfies the equation

$$d\eta/dt + \varepsilon\eta = \varphi(t), \qquad \eta(t_q - 0) = 0. \tag{13}$$

We now fix an arbitrary $t_N > t_q$ and consider the functional

$$J = \int_0^\infty (\xi\mu - \nu\mu^2)\, dt,$$

where

$$\mu(t) = \begin{cases} \eta(t), & t_q \le t < t_N, \\ 0, & t < t_q,\ t \ge t_N; \end{cases} \qquad \xi(t) = -\int_0^t \gamma_\varepsilon(t-\lambda)\mu(\lambda)\,d\lambda. \tag{14}$$

Using Parseval' s equality and condition (8), it is easy to verify that $J \le 0$.

If, from the sequence $t_q, t_{q+1}, \dots, t_N$, we discard the instants t_i for which $|\sigma_i| \le \Delta$, and number the remaining instants by the indices $n_1, n_2, \dots, n_p$, then from the inequality $J \le 0$ it is easy to obtain the estimate, uniform with respect to t_N,

$$\sum_{k=1}^{p-1} J_k \le c_1 = \text{const}, \qquad \text{where } J_k = \int_{t_{n_k}}^{t_{n_{k+1}}} (\sigma\eta - \nu\eta^2)\,dt.$$

Using inequalities (11) and condition (9), one can show that $J_k > c_2$ for sufficiently small $\delta > 0$, where the constant c_2 depends neither on k nor on N. Hence it follows that $p \le c_1/c_2$, i.e., in system (1) $\varphi(t) \equiv 0$ for $t > t_{n_p}$. Passing to the limit as $t \to +\infty$ in equation (1) completes the proof of Theorem 1.

The proof of Theorem 2 is carried out according to the same scheme. First, it is shown recursively that for any $\delta > 0$ there exists such a $t_q > 0$ that, for $t > t_q$, inequalities (11) are satisfied (the values of the quantities r_0 and r_1 are given in the statement of Theorem 2). Then we represent equation (10) in the form

$$\sigma(t) = g(t) - \int_{t_q - 0}^{t} \gamma_\varepsilon(t-\lambda)\eta(\lambda)\,d\lambda - \rho\eta(t) - \varepsilon\xi(t)\rho,$$

where the functions $g(t)$, $\eta(t)$, and $\gamma_\varepsilon(t)$ are defined by formulas (12), (13),

and the function $\xi(t)$ satisfies the equation

$$\frac{d\xi}{dt} = \eta(t), \qquad \xi(t_q) = -\frac{1}{\varepsilon\rho}\left[f_0 - {\sum_{k<q}}' \operatorname{sign}\sigma_k\right]$$

(the prime means that the summation is carried out only over those k for which $|\sigma_k| > \Delta$).

Investigating the functional

$$J = \int_0^\infty [\xi\mu - (\nu+\rho)\mu^2]\,dt,$$

in which the functions ξ and μ are determined by formulas (14), just as in the proof of Theorem 1, we see that, starting from some time, $\varphi(t) \equiv 0$. Passing to the limit as $t \to +\infty$ in equation (15) completes the proof of Theorem 2.

Leningrad State University
named after A. A. Zhdanov

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Note: Figure translations are in progress. See original paper for figures.