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Abstract

Full Text

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UNIFORM CONVERGENCE OF VARIATIONAL-DIFFERENCE SCHEMES

(Presented by Academician V. I. Smirnov, 9 VII 1968)

The estimate of the rate of uniform convergence of positive schemes of the grid method for an elliptic problem coincides with the order of approximation by these schemes of the original equation and boundary conditions on the solution of the problem in question ^(1,2). A certain generalization ⁽³⁾ of the concept of approximation makes it possible to estimate the rate of uniform convergence of the method in the case of discontinuous coefficients. In ⁽³⁾ a method is also indicated for obtaining such estimates on the basis of energy inequalities, using results of V. P. Il' in ⁽⁴⁾; these estimates, however, contain undetermined constants and do not make it possible to trace the increase in the rate of convergence as the smoothness of the given functions increases.

The purpose of the present paper is to single out a class of difference schemes satisfying the maximum principle and, without assuming positivity of the scheme, to obtain estimates of the rate of uniform convergence of variational-difference schemes that make it possible to trace the dependence of the rate of convergence on the smoothness of the given functions.

1°. Consider the Dirichlet problem

$$Au \equiv -\frac{\partial}{\partial x_i} a^{ik} \frac{du}{dx_k} + au = f(x), \quad x \in \Omega; \quad (1)$$

$$u/\partial\Omega = 0, \quad \dim \Omega = n, \quad (2)$$

as the problem of minimizing the functional

$$F(u) = [u, u] - 2(u, f), \quad u \in \dot{W}_2^1(\Omega), \quad (3)$$

where the usual notation ⁽⁵⁾ is adopted, and

$$a^{ik} \xi_i \xi_k \geq \gamma \sum_{i=1}^n |\xi_i|^2, \quad [u, u] \geq \mu_0(u, u).$$

We proceed to the construction of a grid scheme ⁽³⁾ for problem (1)–(2). Let $\{\mathcal{K}\}$ be a family of simplicial complexes $\mathcal{K} \subset \Omega$, depending on a positive parameter h , with

$$c_0 h \leq |l| \leq c_1 h, \quad l \in \mathcal{K}, \quad (4)$$

where $|l|$ is the length of a one-dimensional edge l of the complex \mathcal{K} ; c_0 and c_1 are positive constants independent of h .

Definition. A sequence $\{\mathcal{K}\}$ of simplicial complexes will be called a **sequence of complexes with α -symmetric barycentric stars** if there exists a positive constant c such that the center of symmetry of each of the barycentric stars lies in its $ch^{1+\alpha}$ -neighborhood ($\alpha \geq 0$).

Lemma 1. For every domain Ω with boundary of class $C_{1,\alpha}$ there exists a sequence of complexes with α -symmetric barycentric stars, and

$$\text{mes}(\Omega \setminus |\mathcal{K}|) < c_2 h^{1+\alpha}, \quad c_2 = \text{const} > 0. \quad (5)$$

By \tilde{X} we denote the space of linear extensions $\tilde{u}(x)$ of mesh functions v , defined on the zero skeleton \mathcal{K}_0 of the complex \mathcal{K} and equal to zero on its boundary; moreover, outside the body $|\mathcal{K}|$ of our complex we shall regard the function $\tilde{u}(x)$ as equal to zero. Obviously, $\tilde{X} \subset H_A$, and the problem of minimizing the functional $F(u)$ on the space \tilde{X} leads to the system of equations

$$\tilde{A}\tilde{u} = \tilde{f}, \quad (6)$$

where $[\tilde{A}\tilde{u}, \tilde{u}_1] \equiv [A^{1/2}\tilde{u}, A^{1/2}\tilde{u}_1]$; the latter, in the isomorphic space \bar{X} of mesh functions v , can be written in the form

$$\bar{A}v = \bar{f}. \quad (7)$$

By K, \tilde{K} , and \bar{K} we denote the cones of nonnegative functions in the spaces $L_2(\Omega), \tilde{X}$, and \bar{X} , respectively.

Lemma 2. Suppose that the operator A^{-1} is positive:

$$A^{-1}K \subset K. \quad (8)$$

Then the operators \tilde{A}^{-1} and \bar{A}^{-1} are positive:

$$\tilde{A}^{-1}\tilde{K} \subset \tilde{K}, \quad \bar{A}^{-1}\bar{K} \subset \bar{K}. \quad (9)$$

Remark 1. One says that the maximum principle holds for the operator A if the inclusion (8) is valid. Sufficient conditions for the fulfillment of this principle for the operator A are known (6). It follows from Lemma 2 that there exist nonpositive mesh schemes for which the maximum principle holds.

Theorem 1. Let

$$\partial\Omega \in C_{0,\alpha}, \quad a_{ik} \in C_{2,\alpha}, \quad a \in C_{1,\alpha}, \quad f \in C_{0,\alpha}, \quad (10)$$

and let $\{\mathcal{K}\}$ be a sequence of complexes $\mathcal{K} \subset \Omega$ such that the boundary $\partial|\mathcal{K}|$ of the complex \mathcal{K} lies in an h^α -neighborhood of $\partial\Omega$.

Then the mesh method (6) converges uniformly, and

$$\max_{x \in \Omega} |u^*(x) - \tilde{u}_*(x)| \leq Ch^\alpha, \quad C = \text{const} > 0, \quad (11)$$

where u^*, \tilde{u}_* are the solutions of problems (1)–(2) and (6), respectively.

Theorem 2. Suppose that the conditions

$$\partial\Omega \in C_{1,\alpha}, \quad a_{ik} \in C_{3,\alpha}, \quad a \in C_{2,\alpha}, \quad f \in C_{1,\alpha}, \quad (12)$$

are satisfied, and that $\{\mathcal{K}\}$ is a sequence with α -symmetric barycentric stars, where $\partial|\mathcal{K}|$ lies in a $ch^{1+\alpha}$ -neighborhood of the surface $\partial\Omega$.

Then the rate of uniform convergence of the mesh method is estimated by the inequality

$$\max_{\Omega} |u^* - \tilde{u}_*| \leq Ch^{1+\alpha}. \quad (13)$$

2°. We shall now assume that, in problem (1)–(2), the coefficients are constant.

Theorem 3. Let $f \in W_2^{-1}(\Omega)$, and let the solution u^* of problem (1)–(2) be of class $C_1(\Omega)$.

Then

$$\max_{x \in \Omega} |u - \tilde{u}| \leq C(\omega_{C_1}(u^*, h) + \omega_C(\partial\Omega, h)), \quad (14)$$

where

$$\omega_{C_1}(u^*, h) = \omega_C(u^*, h) + \omega_C(|\partial u^* / \partial x|, h),$$

$$\omega_C(u^*, h) = \sup_{\substack{\|t\| \leq h \\ x \in \bar{\Omega} = \{y | y+t \in \Omega\}}} |u^*(x+t) - u^*(x)|,$$

$$\omega_C(\partial\Omega, h) = \inf_{\substack{\mu \in M \\ \{\varphi_\mu\}}} \omega_C(\varphi_\mu, h);$$

here $\{\varphi_\mu\}$, $\mu \in M$, is an admissible system of local coordinates defining the surface $\partial\Omega$.

Remark 2. Theorems 1-3 remain valid for the non-self-adjoint problem of the form

$$Au + \lambda Tu = f, \quad T = b^i \partial u / \partial x_i, \quad b^i \in C_{.1,\alpha}(\Omega), \quad (15)$$

and in the case $f \equiv 0$ one may regard λ as an eigenvalue.

Here the grid method takes the form (7)

$$\tilde{A}\tilde{u} + \lambda\tilde{T}\tilde{u} = \tilde{f}, \quad \tilde{T} = P_1 T, \quad (16)$$

where P_1 is the orthogonal projector onto the subspace \tilde{X} in $L_2(\Omega)$.

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REFERENCES

1. S. Gerschgorin, *Zs. angew. Math. und Mech.*, **10**, 373 (1930).
2. V. Vazov, Dzh. Forsait, *Difference Methods for Solving Partial Differential Equations*, Moscow, 1963.
3. Yu. K. Dem'yanovich, Candidate dissertation, Voronezh, 1966.
4. V. P. Il'in, *Tr. Matem. inst. im. V. A. Steklova, AN SSSR*, **53** (1959).
5. S. G. Mikhlin, *The Minimum Problem for a Quadratic Functional*, Moscow, 1952.
6. K. Miranda, *Partial Differential Equations of Elliptic Type*, Moscow, 1957.
7. Yu. K. Dem'yanovich, *DAN*, **164**, No. 1 (1965).

Note: Figure translations are in progress. See original paper for figures.

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