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Abstract

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FEATURES OF THE AVERAGING PRINCIPLE FOR EVOLUTION EQUATIONS

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In recent years a number of works have appeared on the application of the Bogolyubov–Krylov method to equations with retarded arguments ($^{1-4}$). It seems to us that such theorems can be obtained in a general scheme, in which an essential role is played by a theorem on the continuous dependence of solutions of a special differential equation on a parameter. This theorem is a generalization of the theorem of M. A. Krasnosel' skii and S. G. Krein (^{5}). It turned out that the proposed scheme covers a broad class of evolution equations of the type of equations with retarded argument.

1. Let Ω_k $(k = 1, \dots, m)$ be certain closed sets on the number line. Let C_i^n $(i = 1, \dots, r;\ r \leq m)$ be spaces of continuous n-dimensional vector-functions $x[s_i]$, defined on Ω_i, $\|x\|_i = \sup_{s_i \in \Omega_i} |x[s_i]|$, and let D_i be sets in C_i^n. Let S_j^n $(j = r+1, \dots, m)$ be spaces of measurable n-dimensional vector-functions $x[s_j]$, defined on Ω_j, $\|x\|_j = \int_{\Omega_j} |x[s_j]| \cdot [1 + |x[s_j]|]^{-1} ds_j$, and let D_j be sets in S_j^n.

On $[0, \infty) \times D_1 \times \dots \times D_m$ an n-dimensional vector-function $F(t, x_1, \dots, x_m)$ is given.

We shall say that $F(t, x_1, \dots, x_m)$ has the average $\bar{F}(x_1, \dots, x_m)$, if

$$\lim_{N \to \infty} N^{-1} \int_0^N F(t, x_1, \dots, x_m)\, dt = \bar{F}(x_1, \dots, x_m) \tag{1}$$

for arbitrary $x_k \in D_k$ $(k = 1, \dots, m)$.

In what follows we shall deal with two families of functions $\chi_k(s_k, t_1, \varepsilon)$ and $\psi_k(s_k, t, \varepsilon)$ $(k = 1, \dots, m)$, $s_k \in \Omega_k$; $t_1 \in [0, T]$, $t \in [0, T/\varepsilon]$; $\varepsilon \in [0, \bar{\varepsilon}]$. The functions $\psi_k(s_k, t, \varepsilon)$ are called generalized times.

We shall say that $\chi_k(s_k, t_1, \varepsilon)$ satisfy condition C for $k = 1, \dots, r$ (respectively, condition S for $k = r+1, \dots, m$), if the function $X_k(t_1, \varepsilon) = \chi_k(s_k, t_1, \varepsilon)$, regarded as a curve in the space $C_k^1(S_k^1)$, as $\varepsilon \to 0$ converges with respect to the measure to a curve $X_k(t_1) = \chi_k^0(s_k, t_1) \geq 0$, which is continuous in t_1 in the space $C_k^1(S_k^1)$.

We shall also say that the generalized times $\psi_k(s_k, t, \varepsilon)$ have averages $\bar{\psi}_k(s_k, t, \varepsilon)$, if the functions

$$\chi_k(s_k, t_1, \varepsilon) = \varepsilon^{-1}\psi_k(s_k, t_1/\varepsilon, \varepsilon) \qquad (t_1 \in [0, T]) \tag{2}$$

satisfy condition C for $k = 1, \dots, r$ or condition S for $k = r+1, \dots, m$, and

$$\bar{\psi}_k(s_k, t, \varepsilon) = \varepsilon^{-1}\chi_k^0(s_k, \varepsilon t) \qquad (t \in [0, T/\varepsilon]). \tag{3}$$

Below, by $R_k(\varepsilon, M, \chi)$ we denote the set of points $\tau \in [0, M]$ for which there exist such $s_k \in \Omega_k$ that $\chi_k(s_k, \tau, \varepsilon) < 0$. The sets $R_k(\varepsilon, M, \psi)$ are defined analogously.

Finally, we shall say that $D_k, \Phi(s_k, \tau, \varepsilon)$ $(0 \le \tau \le L)$ and $\varphi(\tau, \varepsilon)$ $(-l \le \tau \le 0)$ possess the property P_ε in a neighborhood of $z(\tau)$ $(0 \le \tau \le L)$, if $-l \le \Phi(s_k, \tau, \varepsilon) \le \tau$ and for some $\rho > 0$ there is an $\bar{\varepsilon}(\rho) > 0$ such that, for any continuous vector-function $u(\tau)$ $(-l \le \tau \le L)$ which, for some $\varepsilon' \in (0, \bar{\varepsilon}(\rho)]$, satisfies the conditions $u(\tau) = \varphi(\tau, \varepsilon')$ $(-l \le \tau \le 0)$ and $|u(\tau) - z(\tau)| \le \rho$ $(0 \le \tau \le L)$, we have

$$u[\Phi(s_k, \tau, \varepsilon)] \in D_k \qquad (0 < \varepsilon < \bar{\varepsilon}(\rho)).$$

We consider two Cauchy problems

$$dx/dt = \varepsilon F(t, x[\psi_1(s_1, t_1, \varepsilon)], \dots, x[\psi_m(s_m, t, \varepsilon)]), \tag{4}$$

$$x(t) = \varphi(t, \varepsilon), \qquad -h(\varepsilon) \le t \le 0; \tag{5}$$

$$dy/dt = \varepsilon \bar{F}(y[\bar{\psi}_1(s_1, t, \varepsilon)], \dots, y[\bar{\psi}_m(s_m, t, \varepsilon)]), \tag{6}$$

$$y(0) = \varphi(0, 0). \tag{7}$$

Here $h(\varepsilon)$ is any positive number or $+\infty$.

Theorem 1. Suppose that $F(t, x_1, \dots, x_m)$ is continuous in x_k uniformly with respect to the remaining variables, is bounded on the set under consideration, and has the average $\bar{F}(x_1, \dots, x_m)$.

Assume that $\psi_k(s_k, t, \varepsilon)$ have the average $\bar{\psi}_k(s_k, t, \varepsilon)$ and

$$\lim_{\varepsilon \to 0} \varepsilon \operatorname{mes} R_k(\varepsilon, T/\varepsilon, \psi) = \lim_{\varepsilon \to 0} |\varphi(0, \varepsilon) - \varphi(0, 0)| = 0. \tag{8}$$

Suppose, finally, that for $\varepsilon = 1$ problem (6)–(7) has a unique solution $u(t)$, defined on $[0, T]$, and D_k, $\psi_k(s_k, t, \varepsilon)$ $(0 \le t \le T/\varepsilon)$ and $\varphi(t, \varepsilon)$ $(-h(\varepsilon) \le t \le 0)$ possess the property P_ε in a neighborhood of $z(t) = u(\varepsilon t)$ $(0 \le t \le T/\varepsilon)$.

Then for every $\eta > 0$ there is an $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, the solutions $x(t, \varepsilon)$ of problem (4)–(5), defined on $[0, T/\varepsilon]$, differ from the solution $y(t, \varepsilon) = u(\varepsilon t)$ of problem (6)–(7) by less than η on the interval $[0, T/\varepsilon]$, i.e.

$$|x(t, \varepsilon) - y(t, \varepsilon)| < \eta \qquad (0 \le t \le T/\varepsilon). \tag{9}$$

2. For the proof of this assertion we shall need a special theorem on passage to the limit under the integral sign.

We shall say that $X(t_1, x_1, \ldots, x_m, \varepsilon)$ $(t_1, x_1, \ldots, x_m) \in [0, T] \times D_1 \times \cdots \times D_m$ is integrally continuous in ε as $\varepsilon \to 0$, if for any $t_1 \in [0, T]$ the equality

$$\lim_{\varepsilon \to 0} \int_0^{t_1} X(\tau, x_1, \ldots, x_m, \varepsilon)\, d\tau = \int_0^{t_1} X(\tau, x_1, \ldots, x_m, 0)\, d\tau \tag{10}$$

holds.

Theorem 2. Suppose that $X(t_1, x_1, \ldots, x_m, \varepsilon)$ is continuous in $x_k \in D_k$ $(k = 1, \ldots, m)$ uniformly with respect to the other variables, is bounded on the set under consideration, and is integrally continuous in ε as $\varepsilon \to 0$. Let $\chi_k(s_k, t_1, \varepsilon)$, for $k = 1, \ldots, r$, possess property C, and for $k = r + 1, \ldots, m$ possess property S. Suppose that $-k(\varepsilon) \leq \chi_k(s_k, t_1, \varepsilon) \leq t_1$ and

$$\lim_{\varepsilon \to 0} \operatorname{mes} R_k(\varepsilon, T, \chi) = 0. \tag{11}$$

Suppose, furthermore, that a family of continuous vector-functions $u[t_1, \varepsilon]$ $(-k(\varepsilon) \leq t_1 \leq T)$ satisfies the condition

$$\lim_{\varepsilon \to 0} \max_{0 \leq t_1 \leq T} |u[t_1, \varepsilon] - u[t_1, 0]| = 0 \tag{12}$$

and, in addition, $u[\chi_k(s_k, t_1, \varepsilon), \varepsilon]$ and $u[\chi_k^0(s_k, t_1), 0]$ belong to the set D_k.

Then, for any $t_1 \in [0, T]$, the equality holds

$$\lim_{\varepsilon \to 0} \int_0^{t_1} X(\tau, u[\chi_1(s_1, \tau, \varepsilon), \varepsilon], \ldots, u[\chi_m(s_m, \tau, \varepsilon), \varepsilon], \varepsilon)\, d\tau =$$

$$= \int_0^{t_1} X(\tau, u[\chi_1^0(s_1, \tau); 0], \ldots, u[\chi_m^0(s_m, \tau), 0], 0)\, d\tau. \tag{13}$$

From this theorem, in particular, the following assertion follows.

Theorem 3. Let $X(t_1, x_1, \ldots, x_m, \varepsilon)$, $R_k(\varepsilon, T, \chi)$, and $\chi_k(s_k, t_1, \varepsilon)$ satisfy the conditions of Theorem 2, and let the family of functions $g(t_1, \varepsilon)$ $(-k(\varepsilon) \leq t_1 \leq 0)$ satisfy the condition $g(0, \varepsilon) \to g(0, 0)$ as $\varepsilon \to 0$.

Suppose that for $\varepsilon = 0$ the Cauchy problem

$$\frac{du}{dt_1} = X(t_1, u[\chi_1^0(s_1, t_1)], \ldots, u[\chi_m^0(s_m, t_1)], 0), \tag{14}$$

$$u(0) = g(0, 0) \tag{15}$$

has a unique solution $u(t_1)$, defined on $[0, T]$. Let D_k, $\chi_k(s_k, t_1, \varepsilon)$ $(0 \leq t_1 \leq T)$, and $g(t_1, \varepsilon)$ $(-k(\varepsilon) \leq t_1 \leq 0)$ possess the property P_ε in a neighborhood of $u(t_1)$ $(0 \leq t_1 \leq T)$.

Then for any $\eta > 0$ one can indicate an $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, the solutions $v(t, \varepsilon)$, defined on $[0, T]$, of the Cauchy problem

$$dv/dt_1 = X(t_1, v[\chi_1(s_1, t_1, \varepsilon)], \dots, v[\chi_m(s_m, t_1, \varepsilon)], \varepsilon), \tag{16}$$

$$v(t_1) = g(t_1, \varepsilon), \qquad -k(\varepsilon) \leq t_1 \leq 0 \tag{17}$$

satisfy the inequality

$$|v(t_1, \varepsilon) - u(t_1)| < \eta \tag{18}$$

for all $t_1 \in [0, T]$.

If, in addition to the condition of Theorem 3, a local existence theorem holds in a neighborhood of $u(t_1)$, then it can be shown that there is nonlocal continuability of solutions whose initial conditions are close to $u(0) = g(0, 0)$, and these solutions satisfy inequality (18) on the interval $[0, T]$.

3. From Theorem 3 it is easy to obtain Theorem 1. To this end we make the change of time $t_1 = \varepsilon t$. If we set

$$X(t_1, x_1, \dots, x_m, \varepsilon) = F(t_1/\varepsilon, x_1, \dots, x_m) \qquad (\varepsilon \neq 0),$$

$$X(t_1, x_1, \dots, x_m, 0) = \overline{F}(x_1, \dots, x_m); \qquad \chi_k(s_k, t_1, \varepsilon) = \varepsilon \psi_k(s_k, t_1/\varepsilon, \varepsilon),$$

$$\chi_k^0(s_k, t_1) = \lim_{\varepsilon \to 0} \chi_k(s_k, t_1, \varepsilon),$$

$$\overline{\psi}_k(s_k, t, \varepsilon) = \varepsilon^{-1} \chi_k^0(s_k, \varepsilon t); \qquad g(t_1, \varepsilon) = \varphi(t_1/\varepsilon, \varepsilon), \qquad k(\varepsilon) = \varepsilon h(\varepsilon),$$

then problems (4)–(5) and (6)–(7) pass respectively into problems (13)–(14) and (15)–(16); if one takes $x(t, \varepsilon) \equiv v(\varepsilon t)$, and $y(t, \varepsilon) \equiv u(\varepsilon t)$, then the assertion of Theorem 3 passes into the assertion of Theorem 1.

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