Reports of the Academy of Sciences of the USSR

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1969. Volume 187, No. 3

UDC 513.836

MATHEMATICS

D. O. BALADZE

HOMOLOGY K-GROUPS AS GENERALIZED STEENROD GROUPS

(Presented by Academician P. S. Aleksandrov on 20 I 1969)

In the preceding note (^{1}) we introduced generalized Steenrod homology groups, generalizing the classical construction of Steenrod set forth in (^{2}). Here we shall show that the homology K-groups introduced by the author in (^{3}) also fit into the scheme of generalized Steenrod homology groups. From this, in particular, it follows that the homology groups considered by K. A. Sitnikov in the paper (^{4}) are also a special case of generalized Steenrod groups.

By K we shall denote a fixed locally finite complex. Further, if L is an arbitrary locally finite complex, then by L^* we shall denote the barycentric subdivision of the cellular complex $K \times L$, i.e.

$$L^* = (K \times L)'.$$

Let, further, L and M be locally finite complexes and let $f : L \to M$ be some closed simplicial embedding (see (^{1})). Then the map

$$1_K \times f : K \times L \to K \times M,$$

which is cellular, is defined. If we take the barycentric subdivisions of the complexes $K \times L$ and $K \times M$, then we obtain simplicial complexes $L^* = (K \times L)'$ and $M^* = (K \times M)'$, and the map $1_K \times f$ will already be a simplicial map of the first complex into the second. This simplicial map (which, as is easy to see, is a closed simplicial embedding) will be denoted by f^*. Thus, every closed simplicial embedding $f : L \to M$ gives rise to a new closed simplicial embedding $f^* : L^* \to M^*$. The category consisting of all simplicial complexes of the form L^* and their closed simplicial embeddings of the form f^* will be denoted by $\mathfrak{A}$. It is easily verified that conditions 1 and 2 imposed in note (^{1}) on the category $\mathfrak{A}$ are satisfied here.

We now define, for each complex L^* that is an object of the category $\mathfrak{A}$, a certain class of chains $\alpha_p(L^*)$, $p = 0, 1, 2, \ldots$. Namely, let u_{p+r} be an arbitrary (infinite) $(p+r)$-dimensional chain of the locally finite complex L over the group G, and let τ^r be an arbitrary r-dimensional simplex of the complex K; then the barycentric subdivision $(u_{p+r}, \tau^r)^*$ of the chain $u_{p+r} \times \tau^r$ of the locally finite complex $K \times L$ is a chain of the complex $L^* = (K \times L)'$. We shall include all possible chains of the form

$$\sum_{\substack{\tau^r \in K \\ r=0,1,2,\ldots}} (u_{p+r}, \tau^r)^*$$

in $\alpha_p(L^*)$.

We define the boundary operator $d : \alpha_p(L^*) \to \alpha_{p-1}(L^*)$ in accordance with the boundary operator in (^{3}). Namely, we set

$$d(u_{p+r}, \tau^r)^* = (\partial u_{p+r}, \tau^r)^* + (-1)^{p+r+1}(u_{p+r}, \delta\tau^r)^*, \tag{1}$$

where ∂ is the boundary operator in L, and δ is the coboundary operator in K. From this formula it follows easily that $d \circ d = 0$. Further, it is easy to see that the formula

(1) can be rewritten in the following equivalent form:

$$\begin{aligned} d\left(\sum_\tau (x_\tau, \tau)^*\right) &= \sum_\tau (\partial(x_\tau), \tau)^* - \sum_\tau \sum_\sigma (-1)^{\dim x_\sigma} [\tau : \sigma](x_\sigma, \tau)^* \\ &= \sum_\tau (\partial(x_\tau), \tau)^* + \sum_\tau (-1)^{\dim x_\tau}(x_{\partial\tau}, \tau)^*, \end{aligned} \tag{2}$$

where τ ranges over all (arbitrarily oriented) cells of the complex K; further, σ ranges over cells related to τ by the relation $\dim\sigma = \dim\tau - 1$, and $[\tau : \sigma]$ denotes the incidence coefficient of the cells τ and σ in the complex K. Formula (2) shows that the operator d corresponds to the boundary operator defined in note (^{3}). More precisely, let $x = \{x_\tau\}$ be a collection of chains of the complex L, indexed by the cells $\tau \in K$, such that $\dim x_\tau = p + \dim\tau$, where p is a fixed nonnegative integer. In other words, x is a p-dimensional K-chain of the complex L in the sense of note (^{3}). By $\partial x = \{(\partial x)_\tau\}$ we denote the boundary of this K-chain x (in the sense of (^{3})). Then, as formula (2) shows,

$$d\left(\sum_\tau (x_\tau, \tau)^*\right) = \sum_\tau ((\partial x)_\tau, \tau)^*, \tag{3}$$

which establishes the correspondence between the operator

$$d : \alpha_p(L^*) \to \alpha_{p-1}(L^*)$$

and the operator

$$\partial : C_p^K(L, G) \to C_{p-1}^K(L, G).$$

Thus we have a category $\mathfrak{A}$, consisting of simplicial complexes of a certain kind (namely, complexes of the form $L^* = (K \times L)'$) and their closed simplicial embeddings of a certain kind (namely, maps of the form f^*). Further, for every complex L^* that is an object of the category $\mathfrak{A}$, we have a certain class of chains $\alpha(L^*)$ (and, as is easy to check, the conditions 1-4 imposed on the class of chains in note (^{1}) are satisfied here). Therefore we can define the $(\mathfrak{A}, \alpha)$-homology groups of an arbitrary compactum Φ, as indicated in note (^{1}).

Theorem. The homology groups constructed here

$$H_p^{(\mathfrak{A},\alpha)}(\Phi, G)$$

are naturally isomorphic to the K-homology groups of the compactum Φ over the coefficient group G:

$$H_p^{(\mathfrak{A},\alpha)}(\Phi, G) \simeq \Delta_p^K(\Phi, G).$$

The proof of this theorem is carried out according to the following scheme. Let $L^* = (K \times L)'$ be a complex belonging to the category $\mathfrak{A}$, and let $\varphi : (L^*)^0 \to \Phi$ be a regular mapping of its zero-dimensional skeleton into the metric compact space Φ. Further, let $u \in \alpha_p(L^*)$, so that the triple (L^*, φ, u) is a p-dimensional regular chain of the space Φ over the coefficient group G. The chain u can be written in the form

$$u = \sum_{\tau \in K} (u_\tau, \tau)^*,$$

where u_τ is some chain of the complex L, with $\dim u_\tau = p + \dim \tau$. Now let $\varphi_\tau : L^0 \to \Phi$ be the mapping of the zero-dimensional skeleton of the complex L into Φ, defined by the relation

$$\varphi_\tau(a) = \varphi(o_\tau \times a), \qquad a \in L^0,$$

where o_τ is the center of the simplex τ. The mapping φ_τ carries each simplex of L into some skeleton of the space Φ. As a result, the chain u_τ passes into

some chain of nerves of the space Φ, which we shall denote by x_τ. It is readily verified that the degrees of fineness of the chains x_τ decrease without bound when τ "goes to infinity" in the complex K, i.e., $x = \{x_\tau\}$ is a p-dimensional K-chain of the space Φ over the group G. The correspondence thus constructed

$$(L^*, \varphi, u) \to \{x_\tau\}$$

between regular p-dimensional chains of the space Φ and p-dimensional K-chains of the space Φ is precisely what establishes the isomorphism indicated in the theorem. Formulas (1), (2), (3) give the algebraic basis for verifying that this mapping is an isomorphism.

V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
27 XII 1968

CITED LITERATURE

[1] D. O. Baladze, DAN, **186**, No. 5 (1969).
[2] N. E. Steenrod, UMN, **2** (18), issue 2, 56 (1947).
[3] D. O. Baladze, Soobshch. AN GruzSSR, **52**, No. 2, 283 (1968).
[4] K. A. Sitnikov, Matem. sborn., **34** (76), No. 1, 3 (1954).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.