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Fig. 1

Figure 1: Fig. 1

## Abstract

## Full Text

## THEORY OF ELASTICITY

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# EQUIVALENT STIFFNESS OF A DOUBLY PERIODIC LATTICE REINFORCED BY ELASTIC RINGS

The problem of reducing a uniformly perforated plate to an equivalent solid plate, in the sense of tensile (or flexural) stiffness, has been considered, in one formulation or another, by various authors (see, for example, ([1])).

In the present paper a solution of the indicated problem is given for the case of isotropic tension of an infinite plate with a doubly periodic system of identical circular holes reinforced by thin elastic rings made of a material different from that of the lattice. In solving the problem posed we start from the solution of the corresponding doubly periodic problem in its exact formulation; however, the ring is assumed to work only in tension. The results will be valid if the lattice does not differ greatly from a regular one and the load at infinity is close to isotropic.

**Fig. 1**

1. Consider a symmetric doubly periodic lattice, stretched in two directions, with thin elastic rings fitted into the holes (Fig. 1). Let $E$ and $\mu$ be the Young' s modulus and Poisson' s ratio of the plate material, and $E_1$ and $\mu_1$ the corresponding elastic constants of the ring material. We shall assume that the ring works only in tension.

The equilibrium conditions for an element of the ring in the normal and tangential directions to the contour of the hole give

$$\sigma_r bt = F\sigma, \qquad F\,\partial\sigma/\partial\theta = -\tau_{r\theta}bt, \tag{1.1}$$

where $\sigma_r$ and $\tau_{r\theta}$ are the normal and tangential stress components in the lattice along the contour of the layer; $\sigma$ is the normal stress in the transverse section of the ring; $F$ is the cross-sectional area of the ring; $b$ and $t$ are the radius of the hole and the thickness of the plate.

The condition of compatibility of deformation along the line of bonding of the ring with the lattice has the form

$$\sigma/E_1 = (\sigma_\theta - \mu\sigma_r)/E. \tag{1.2}$$

Eliminating the quantity $\sigma$ from relations (1.1) and (1.2), we arrive at the boundary conditions for the components $\sigma_r$ and $\sigma_\theta$ acting along the contour of the hole,

$$\sigma_r\left(\frac{E}{E_1}\frac{bt}{F} + \mu\right) = \sigma_\theta, \qquad \frac{\partial\sigma_r}{\partial\theta} = -\tau_{r\theta}. \tag{1.3}$$

The last two conditions may be reduced to a single condition in complex form

$$\sigma_r - i\tau_{r\theta} = \frac{1}{\gamma}\sigma_\theta + i\frac{\partial\sigma_r}{\partial\theta}, \qquad \gamma = \frac{Ebt}{E_1F} + \mu. \tag{1.4}$$

According to (2),

$$\sigma_r - i\tau_{r\theta} = \Phi_s(\tau) + \overline{\Phi_s(\tau)} - \{\bar{\tau}\Phi_s'(\tau) + \Psi_s(\tau)\}e^{2i\theta}, \tag{1.5}$$

where $\Phi_s(z) = (\sigma_1 + \sigma_2)/4 + \Phi(z)$; $\Psi_s(z) = (\sigma_2 - \sigma_1)/2 + \Psi(z)$; $\sigma_1 = \sigma_x^{(\infty)}$; $\sigma_2 = \sigma_y^{(\infty)}$; $\Phi(z)$, $\Psi(z)$ are functions regular in the region occupied by the lattice; $\tau \in L$; $L$ is the boundary of the region; $\theta$ is the angle between the normal to $L$ and the $x$-axis.

Representations of the functions $\Phi$ and $\Psi$, reflecting the doubly periodic character of the problem, are taken, according to (3), in the form

$$\Phi(z) = a_0 + \sum_{k=0}^{\infty} \frac{a_{2k+2}\lambda^{2k+2}}{(2k+1)!}\rho^{2k}(z), \tag{1.6}$$

$$\Psi(z) = \beta_0 + \sum_{k=0}^{\infty} \beta_{2k+2}\frac{\lambda^{2k+2}\rho^{2k}(z)}{(2k+1)!} - \sum_{k=0}^{\infty} a_{2k+2}\frac{\lambda^{2k+2}Q^{2k+1}(z)}{(2k+1)!},$$

where $\rho(z)$ and $Q(z)$ are special meromorphic functions; $\lambda$ is the dimensionless radius of the hole; $a_{2k+2}$ and $\beta_{2k+2}$ are constants to be determined from the boundary-value problem.

The constants $a_0$ and $\beta_0$ are determined from the static conditions "at infinity." They have the form (3)

$$K_0 = \frac{\delta_1}{\omega_1} + \frac{2\pi i}{\omega_1 \omega_2 - \omega_2 \omega_1}, \qquad K_1 = \frac{\pi i}{\omega_2 \omega_1 - \omega_1 \omega_2}, \qquad K_2 = \frac{\gamma_1 - \delta_1}{\omega_1} - \frac{4\pi i}{\omega_1 \omega_2 - \omega_1 \omega_2}. \tag{1.7}$$

Here $\omega_1$ and $\omega_2$ are the fundamental periods of the lattice; $\omega_1 = 2$; $\omega_2 = 2le^{i\alpha}$; $\delta_1 = 2\zeta(\omega_1/2)$; $\gamma_1 = 2Q(\omega_1/2) - \bar{\omega}_1 \rho(\omega_1/2)$; $\zeta(z)$ is the Weierstrass zeta-function.

Substituting the expansions of the representations (1.6) in a neighborhood of zero into the boundary condition (1.4), and taking (1.5) into account, we arrive at an infinite system of linear algebraic equations

$$\alpha_{2j+2} = \sum_{k=0}^{\infty} a_{j,k} \alpha_{2k+2} + b_j, \tag{1.8}$$

where

$$a_{j,k} = \frac{(2j+1)\lambda^{2j+2k+2}}{2j+2-(2j+1)\Omega} \gamma_{j,k};$$

$$\gamma_{0,0} = \frac{3g_2\lambda^2}{8}\Omega + K_2 + \frac{2K_0^2\lambda^2\Omega}{1-2K_1\lambda^2\Omega} + \sum_{i=1}^{\infty} \frac{(2i+1)g_{i+1}^2\lambda^{4i+2}}{2^{4i+4}}[(2i+1)\Omega - 2i];$$

$$\gamma_{0,k} = \gamma_{k,0} = \frac{(2k+3)(2k+4)g_{k+2}\lambda^2}{2^{2k+5}}\Omega - \frac{(2k+2)\rho_{k+1}}{2^{2k+2}} + \frac{2K_0\lambda^2\Omega}{1-2K_1\lambda^2\Omega}\frac{g_{k+1}}{2^{2k+2}} +$$

$$+ \sum_{i=1}^{\infty} \frac{(2i+2k+1)!\, g_{i+1} g_{i+k+1} \lambda^{4i+2}}{(2i)!(2k+1)!\, 2^{4i+2k+4}}[(2i+1)\Omega - 2i];$$

$$\gamma_{j,k} = \gamma_{k,j} = \frac{(2j+2k+4)!\, g_{j+k+2}\lambda^2}{(2j+2)!(2k+2)!\, 2^{2j+2k+4}}\Omega - \frac{(2j+2k+2)!\rho_{j+k+1}}{(2j+1)!(2k+1)!\, 2^{2j+2k+2}} +$$

$$+ \frac{2\lambda^2\Omega\, g_{j+1} g_{k+1}}{(1-2K_1\lambda^2\Omega)\, 2^{2j+2k+4}} +$$

$$+ \sum_{i=1}^{\infty} \frac{(2j+2i+1)!(2i+2k+1)!\, g_{j+i+1} g_{k+i+1} \lambda^{4i+2}}{(2j+1)!(2k+1)!(2i+1)!(2i)!\, 2^{2j+2k+4i+4}}[(2i+1)\Omega - 2i];$$

$$b_0 = \frac{\sigma_2 - \sigma_1}{2} + \frac{\sigma_1 + \sigma_2}{2} \frac{K_3\lambda^2\Omega}{1-2K_1\lambda^2\Omega};$$

Fig. 2

Figure 2: Fig. 2

$$b_j = \frac{(\sigma_1+\sigma_2)\lambda^{2j+2}(2j+1)\Omega}{2^{2j+3}(1-2K_1\lambda^2\Omega)}, \qquad j = 1, 2, \ldots;$$

$$g_i = \sum_{m,n}{}' \frac{1}{T^{2i}}; \qquad \rho_i = \sum_{m,n}{}' \frac{\overline{T}}{T^{2i+1}}; \qquad T = m + nle^{i\alpha}.$$

The quantity $\Omega$ is determined by the formula

$$\Omega = (\gamma-1)/(\gamma+1). \tag{1.9}$$

Since $\gamma > 0$ and varies within the limits $\mu < \gamma < \infty$, for $\Omega$ the double inequality holds

$$(\mu-1)/(\mu+1) < \Omega < 1. \tag{1.10}$$

We shall call the quantity $\Omega$ the reinforcement parameter.

**Fig. 2**

The coefficients $\beta_{2k+2}$ are expressed in terms of $\alpha_{2k+2}$ by the formulas

$$\beta_2 = \frac{\Omega}{1-2K_1\lambda^2\Omega}\left\{\frac{\sigma_1+\sigma_2}{2} + 2K_0\lambda^2\alpha_2 + 2\sum_{k=1}^{\infty}\alpha_{2k+2}\frac{g_{k+1}\lambda^{2k+2}}{2^{2k+2}}\right\}; \tag{1.11}$$

$$\beta_{2j+4} = (2j+3)\Omega\alpha_{2j+2} + \sum_{k=0}^{\infty}\alpha_{2k+2}\frac{(2j+2k+3)!\,g_{j+k+2}\lambda^{2j+2k+4}}{(2j+2)!(2k+1)!\,2^{2j+2k+4}}\left[(2j+3)\Omega-(2j+2)\right] \quad (j = 0, 1, \ldots).$$

**2.** The results of calculations of the reduced elastic moduli for regular lattices are given in Fig. 2. In Fig. 2a the diagrams of the reduced elastic modulus under uniform tension are given as functions of $\lambda$ (the ratio of the hole diameter to the pitch) for various values of the reinforcement parameter $\Omega$. In Fig. 2b the variation of the same quantity is shown for a square network of holes. In the calculations it was assumed that $\mu = 0.3$.

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*Note: Figure translations are in progress. See original paper for figures.*

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