ON THE STRONG MIXING PROPERTY FOR MARKOV CHAINS WITH A COUNTABLE NUMBER OF STATES

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Abstract

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ON THE STRONG MIXING PROPERTY FOR MARKOV CHAINS WITH A COUNTABLE NUMBER OF STATES

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Let $\{X_j\}, j = 0, 1, \dots,$ be a random process; $\mathfrak{M}_a^b$ the σ-algebra of events generated by the variables $\{X_a, \dots, X_b\}$. If, as $k \to \infty$,

$$\sup_n \sup_{\substack{A\in\mathfrak{M}_0^n \\ B\in\mathfrak{M}_{n+k}^\infty}} |P(AB) - P(A)P(B)| = \alpha(k) \to 0, \tag{1}$$

then the process X_j is said to satisfy the condition of strong mixing (s.m.) with coefficient $\alpha(k)$. Condition (1) was introduced by M. Rosenblatt (5). The basic properties of processes for which (1) holds, under various restrictions, were studied in the works of V. A. Volkonskii and Yu. A. Rozanov (1), I. A. Ibragimov (2), and others. In the present work it is proved that a broad class of Markov chains possesses property (1); in addition, some examples of processes with s.m. are given.

1. We shall consider homogeneous Markov chains $\{X_j\}$ with a countable set of states $I = \{i\}$, transition probability matrix $P = (p_{ij})$, and initial distribution $p = (p_i)$, forming one recurrent positive class with period $d = 1$. Under these assumptions the following is true.

Theorem 1. *The process X_j has the s.m. property.*

The proof is based on the following elementary verifiable lemma:

Lemma 1. *Let $P_{ij}^{(n)} = P\{X_n = j \mid X_0 = i\}$; $P_j^{(n)} = P\{X_n = j\}$;*

$$\beta(k) = \sup_n \sup_{\substack{A\subset\Gamma \\ B\subset\Gamma}} \left| \sum_{\substack{i\in A \\ j\in B}} P_i^{(n)} \left(P_{ij}^{(k)} - P_j^{(n+k)} \right) \right| .$$

Then

$$\beta(k) \leqslant \alpha(k) \leqslant 2\beta(k).$$

Example. In this example we shall show that if the recurrent class is null, then s.m. need not hold. Let I be the set of all integers;

$$p_0 = 1; \quad p_{00} = 0, \quad p_{0i} = p_{0,-i} = \alpha_i; \quad i = 1, 2, \dots; \quad 0 < \alpha_i, \quad \sum_1^\infty \alpha_i = \tfrac{1}{2};$$

$p_{i,i-1} = 1$ for $i > 0$ and $p_{i,i+1} = 1$ for $i < 0$. In this case $f_{00}^{(n)} = 2\alpha_{n-1}$ and, consequently, the chain is recurrent. If the class is null, then it is easily verified that $\alpha(n) \geqslant \frac{1}{4}$, and thus the s.m. condition is not fulfilled.

In connection with Lemma 1, the following theorem is of interest; it is a strengthening of Feller' s results (^{6}).

Theorem 2. *Denote by* $f_n = f_{jj}^{(n)}$ *the probability, upon leaving state* j*, of returning to* j *for the first time at the* n*-th step, and let* $P_n = P_{jj}^n$, $s > 1$*. Then:*

1) *in order that* $|P_n - P_{n-1}| = O(1/n^s)$*, it is necessary and sufficient that* $f_n = O(1/n^{s+1})$*;*
2) in order that

$$\sum_1^\infty n^s |P_n - P_{n-1}| < \infty,$$

it is necessary and sufficient that

$$\sum_0^\infty n^{s+1} f_n < \infty.$$

The proof is similar to Feller' s proof and is based on certain results from the theory of normed rings.

Corollary. If $f_n = O(1/n^{s+1})$, then $|P_n - \pi| = O(1/n^{s-1})$; if

$$\sum n^{s+1} f_n < \infty,$$

then

$$\sum n^{s-1} |P_n - \pi| < \infty.$$

Here $\pi = \lim P_n$, $s > 1$.

We formulate one more theorem, which will be needed below.

Theorem 3. Let $\{X_n\}$ be a Markov chain with the s.m. property, satisfying the assumptions indicated above; let $p = (p_j)$ be the initial distribution. Suppose that the normalized sums

$$Z_n = (S_n - A_n)/B_n,$$

where

$$S_n = \sum_0^n X_k,$$

converge in distribution to $F(x)$. Then this assertion remains valid if the initial distribution is replaced by any other one.

2. The following examples show that for processes with the s.m. property the central limit theorem need not hold, and that the variance of partial sums need not have the form $nh(n)$, where $h(n)$ is a slowly varying function.

Example 1 (Döblin's example, see (^{3})). Let I be the set of nonnegative integers; $a_0 = 1,\ 0 < a_n < 1$ if $n \geq 1$; $p_0 = 1$, and

$$p_{n,0} = 1 - a_n, \quad p_{n,n+1} = a_n, \quad n \geq 0.$$

Let $\beta_0 = 1,\ \beta_n = \prod_0^{n-1} a_k$. All states form a class with period 1, and in this case

$$f_{00}^{(1)} = 0, \quad f_{00}^{(n)} = \beta_{n-1} - \beta_n, \quad n \geq 2.$$

Put

$$f_{00}^{(n)} = c/n^3 \ln^2 n, \quad n \geq 2.$$

From this the a_n are determined for $n \geq 1$. With this choice of $f_{00}^{(n)}$ the chain will be recurrent and positive. Define a new Markov chain: fix $r > 2$ and take as I_1 the set $\{j^{1/r} \mid j \in I\}$; leave the transition-probability matrix as before, and take the stationary distribution as the initial distribution. Let X_n be the state of the chain at the n-th moment of time. Then, using Theorems 1, 2, and 3, one can verify that the process X_n will be a stationary process in the narrow sense satisfying the s.m. condition with coefficient $\alpha(n)$, $\sum \alpha(n) < \infty$, for which: 1) $E|X_j|^r < \infty$; 2) DS_n grows no more slowly than $n^{1+(1/r)}h(n)$, where $h(n)$ is a slowly varying function; 3) for no choice of constants A_n and $B_n > 0$ does the quantity

$$Z_n = (S_n - A_n)/B_n$$

converge in distribution to a stable law with exponent

$$\alpha = 2r/(r+1).$$

Example 2. In this example the transition-probability matrix is constructed in the same way as in Example 1. The states are the numbers

$$\left\{1 + j\frac{1}{k} + 1\right\},$$

and the numbers α_j are determined from the relation

$$f_{00}^{(n)} = c/n^k,$$

where $n \geq 2,\ 1+\sqrt{2} < k \leq 3$. The initial distribution is stationary. Here the process $\{X_j\}$ will be stationary in the narrow sense, and $|X_j| \leq 2$ with probability 1,

$$DS_n \geq c_1 n^{2/(k-1)} h(n),$$

where $h(n)$ is a slowly varying function. The mixing coefficient is estimated with the help of Theorem 2:

$$c_2 n^{2-k} \leq \alpha(n) \leq c_3 n^{2-k}.$$

Moreover, for a certain choice of constants the quantity Z_n is asymptotically stable with exponent

$$\alpha = k - 1.$$

The case $k = 3$ is of interest for comparison with Theorem 18.5.4 (^{4}). There, under the condition $\sum \alpha(n) < \infty$ and under the assumption that $|X_j|$ is bounded with probability 1, the central limit theorem is proved. In our example, $|X_j| \leq 2$ with probability 1, the central limit theorem holds, but

$$\sum \alpha(n) = \infty.$$

3. We now give a necessary and sufficient condition for a homogeneous Markov chain with a countable number of states to possess the property of uniformly strong mixing (u.s.m.). (For the u.s.m. condition, see (4).)

Define $\varphi(n)$ by the equality

$$\varphi(n) = 1 - \frac{1}{2} \sup_{i,j \in I} \sum_{k \in I} \left| P_{ik}^{(n)} - P_{jk}^{(n)} \right|.$$

$\varphi(n)$ is the ergodicity coefficient introduced by Dobrushin (7).

The following assertions hold:

1. For a homogeneous chain, either $\varphi(n) \equiv 0$, or $\varphi(n) \to 1$ exponentially fast.
2. A homogeneous chain satisfies the u.s.m. condition if and only if $\varphi(n) \to 1$.

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