ON THE EIGENFUNCTIONS OF THE FIRST BOUNDARY-VALUE PROBLEM FOR ELLIPTIC EQUATIONS DEGENERATING AT POINTS

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Abstract

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ON THE EIGENFUNCTIONS OF THE FIRST BOUNDARY-VALUE PROBLEM FOR ELLIPTIC EQUATIONS DEGENERATING AT POINTS

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Works ([1–6]), etc., are devoted to equations of elliptic type admitting degeneration, or, in other terminology ([1]), to equations with singular coefficients. A detailed survey of the principal results on degenerate elliptic equations is contained in the monograph of M. M. Smirnov ([5]). In the present article a method is indicated for finding the eigenfunctions of elliptic equations degenerating at interior points, and some cases of completeness of the system of functions so obtained are given.

Let E_n be Euclidean space of n dimensions. Denote by $r(x)$ the distance from the point $x = (x_1, x_2, \dots, x_n)$ to the origin O. The boundary of an arbitrary set A of points of the space E_n will be denoted by ΓA. By K_ρ we shall denote the n-dimensional open ball of radius ρ with center at the point O.

Let a bounded closed domain G of class $C_{(2,\alpha)}$ (see ([9])) be given, and let the origin O be an interior point of G. Denote by G_0 the domain $G - \Gamma G - O$. Consider in G the elliptic differential equation

$$Lu \equiv -\frac{\partial}{\partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j}\right) + C(x)u(x) = \lambda\sigma(x)u(x). \qquad (1)$$

Here and throughout what follows, summation from 1 to n over pairs of equal indices is understood. We assume that $A_{ij}(x) = A_{ji}(x)$ and that the quadratic form $A_{ij}\xi_i\xi_j$ is positive definite in any closed domain $G_\rho = G - K_\rho$, where ρ is an arbitrary small positive number. The coefficients $A_{ij}(x) \in C_{(1,\alpha)}(G - O)$, while $C(x) > 0$, $\sigma(x) > 0$ belong to the class $C_{(0,\alpha)}$ in $G - O$. The function $\sigma(x)$ is summable in G, and the relation

$$\lim_{r(x)\to 0}\frac{\sigma(x)}{C(x)} = 0 \qquad (2)$$

is satisfied.

Denote by $\mathcal{L}_2(A, \sigma)$ the Hilbert space of measurable functions whose squares are integrable over the domain A with weight $\sigma(x)$.

Suppose there exists a twice continuously differentiable and positive function $w(x)$ in $K_d - O$, tending to infinity as $r(x) \to 0$, and such that

$$Lw > 0 \quad \text{in } K_d, \tag{3}$$

where d is some positive number. Then the differential expression Lu will be called w-normal (see ([6])).

Let us note some particular cases in which condition (3) is fulfilled for $w(x) = r^{-\beta}(x)$, $\beta > 0$. Let the coefficients $A_{ij}(x)$ have the form $A_{ij} = r^{\mu}(x)a_{ij}(x)$, where $a_{ij}(x) \in C_{(1,0)}(G)$, and

$$a_{ij}(0) = \begin{cases} 1, & \text{for } i = j = 1, 2, \dots, m; \quad 0 \le m \le n, \\ 0, & \text{for the remaining } i, j = 1, 2, \dots, n. \end{cases} \tag{4}$$

Then condition (3) will be satisfied if there exists a positive β such that in some ball K_d the inequality

$$C(x) + \beta(\mu + m - \beta - 2)r^{\mu-2}(x) \ge \tau r^{\beta}(x), \tag{5}$$

holds, where τ is some positive number.

Let, for any positive number λ, the differential expression

$$L_\lambda u \equiv Lu - \lambda\sigma u$$

be w_λ-normal in the ball $K_{d(\lambda)}$, where, on the basis of condition (2),

$$C(x) - \lambda\sigma(x) \ge 0.$$

Consider the problem of eigenvalues and eigenfunctions for the differential expression Lu.

Find those values of the parameter λ for which there exist functions $u(x) \not\equiv 0$, bounded in G, $u(x) \in C_{(2,0)}(G_0)$, satisfying equation (1) in G_0, continuous in $G - O$, and subject to the condition

$$u\big|_{\Gamma_G} = 0. \tag{6}$$

We shall call these values λ eigenvalues, and the corresponding nontrivial solutions $u(x)$ eigenfunctions.

The solution of problem (1), (6) is carried out by a certain limiting transition from the solutions of the following problem.

Find those values λ_ε for which there exist functions $u_\varepsilon(x) \not\equiv 0$, continuous in G_ε, $0 < \varepsilon < d$, belonging to $C_{(2,0)}$ in the domain $G_\varepsilon - \Gamma G_\varepsilon$, satisfying there the equation

$$Lu_\varepsilon(x) = \lambda_\varepsilon \sigma(x) u_\varepsilon(x), \tag{7_ε}$$

and subject to conditions (6) and

$$u_\varepsilon(x)\big|_{\Gamma K_\varepsilon} = 0. \tag{8_ε}$$

Problem (7_ε), (6) and (8_ε) is, under our assumptions, regular and has been studied in detail (see ([7–9])).

For any fixed ε there exists a countable sequence, nondecreasing with increasing index, of eigenvalues $\lambda_\varepsilon^{(k)} > 0$, with the only limiting point at infinity, and the corresponding system of eigenfunctions $\{u_\varepsilon^{(k)}(x)\}$, $u_\varepsilon^{(k)}(x) \in C_{(2,\alpha)}(G_\varepsilon)$ (see ([7–9])). This system of functions is complete and orthonormal in $\mathcal{L}_2(G_\varepsilon, \sigma)$.

From the minimax principle (see ([8])) it follows that the k-th eigenvalues $\lambda_\varepsilon^{(k)}$ do not increase under a monotone decrease of ε to zero. Therefore the limit exists

$$\lim_{\varepsilon \to 0} \lambda_\varepsilon^{(k)} = \lambda^{(k)}. \tag{9_k}$$

It is proved that these $\lambda^{(k)}$ are eigenvalues of problem (1), (6), corresponding to eigenfunctions $u^{(k)}(x)$. This eigenfunction $u^{(k)}(x)$ can be obtained as the limit, converging in $G - O$ together with its first and second derivatives, of the sequence $u_{\varepsilon_k}^{(k)}(x)$, $\{\varepsilon_k\} \subset \{\varepsilon_{k-1}\}$, of the k-th eigenfunctions of problem (7_ε), (6), (8_ε), $\varepsilon = \varepsilon_k$.

Theorem 1. *Let the closed domain G belong to the class $C_{(2,\alpha)}$, the coefficients $A_{ij}(x) \in C_{(1,\alpha)}(G - O)$, the functions $C(x)$ and $\sigma(x)$ be positive and belong to the class $C_{(0,\alpha)}$ in $G - O$, $\sigma(x)$ be summable in G, and let relation (2) hold. Let, for any positive number λ, the differential expression $L_\lambda u$ be w_λ-normal in the ball $K_{d(\lambda)}$.*

Then there exists a countable sequence, nondecreasing with increasing index, of positive eigenvalues $\lambda^{(k)}$, defined by equality (9_k), with the only limiting point at infinity, and the corresponding system of eigenfunctions $u^{(k)}(x)$ of problem (1), (6) is complete and orthonormal in the Hilbert space $\mathcal{L}_2(G, \sigma)$.

Let L denote the symmetric operator, extended in the sense of Friedrichs (see ([3,5,7])), determined by the differential expression Lu on the set of functions twice continuously differentiable in G, finite-

finite at the origin and on the boundary of the domain G. Theorem 1 makes it possible to formulate the following result.

Theorem 2. *If all the conditions of Theorem 1 are satisfied, then the operator L will be self-adjoint, positive definite, and have a discrete spectrum in the space $\mathcal{L}_2(G,\sigma)$.*

2. Consider the elliptic differential equation (1) in the whole space E_n. Denote by $K(a,b)$ the domain of points x satisfying the inequalities $a \leq r(x) \leq b$. Suppose that $A_{ij} = A_{ji}$, and that the quadratic form $A_{ij}\xi_i\xi_j$ is positive definite in any closed domain $K(\rho,\rho^{-1})$, where ρ is an arbitrary small positive number. Analogously to the preceding case, we consider the problem of eigenfunctions $u(x)$, bounded in the whole space E_n, belonging to $C_{(2,0)}$ and satisfying equation (1) in any domain $K(\delta,\delta^{-1})$, where δ is an arbitrary small positive number. In this case the following results hold.

Theorem 3. *Let the coefficients of the differential expression $L_\lambda u$ in any closed domain $K(\rho,\rho^{-1})$ satisfy the conditions: $A_{ij}(x) \in C_{(1,\alpha)}$, $C(x)$ and $\sigma(x)$ are positive and belong to the class $C_{(0,\alpha)}$. Let $\sigma(x)$ be summable in E_n, let condition (2) be satisfied, and let the analogous condition at infinity be satisfied*

$$\lim_{r(x)\to\infty} \frac{\sigma(x)}{C(x)} = 0. \tag{2'}$$

Let, for any positive number λ, the expression $L_\lambda u$ be w_λ-normal in $K_{d(\lambda)}$ and, analogously, W'_λ-normal outside the ball $K_{d'(\lambda)}$, where $d'(\lambda)$ is some positive number.

Then there exists a countable, nondecreasing with increasing index, sequence of positive eigenvalues $\lambda^{(k)}$, with the only limit point at infinity, and the corresponding system of eigenfunctions $w^{(k)}(x)$ of equation (1) in E_n is complete and orthonormal in the Hilbert space $\mathcal{L}_2(E_n,\sigma)$.

If, as in the preceding paragraph, we denote by L' the Friedrichs extension of the symmetric operator defined by the differential expression Lu on the set of functions twice continuously differentiable in E_n, finite at the origin and at infinity, then the following is valid.

Theorem 4. *If all the conditions of Theorem 3 are satisfied, then the operator L' will be self-adjoint, positive definite, and have a discrete spectrum in the space $\mathcal{L}_2(G,\sigma)$.*

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CITED LITERATURE

1. L. G. Mikhailov, *A new class of singular integral equations and its applications to differential equations with singular coefficients*, Dushanbe, 1963.

2. M. V. Keldysh, DAN, **77**, No. 2 (1951).

3. S. G. Mikhlin, Vestn. Leningrad. Univ., **3**, No. 8, 19 (1954).

4. O. A. Oleinik, Mat. sborn., **69** (111), 1, 111 (1966).

5. M. M. Smirnov, *Degenerate Elliptic and Hyperbolic Equations*, Moscow, 1966.

6. A. I. Achil' diev, DAN, **152**, No. 1, 13 (1963).

7. S. G. Mikhlin, *The Minimum Problem of a Quadratic Functional*, 1952.

8. S. G. Mikhlin, *Variational Methods in Mathematical Physics*, 1957.

9. O. A. Ladyzhenskaya, N. N. Ural' tseva, *Linear and Quasilinear Equations of Elliptic Type*, 1964.

Note: Figure translations are in progress. See original paper for figures.