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TRACE FORMULA FOR A SYSTEM OF THREE PARTICLES

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1. Notation. Let $R(z)$ be the resolvent of the quantum-mechanical energy operator H of a system of three pairwise interacting nonrelativistic particles with the motion of the center of inertia separated out. Let $R_0(z)$ be the resolvent of the kinetic-energy operator H_0, and let $R_\alpha(z)$ $(\alpha = 1, 2, 3)$ be the resolvent of the operator H_α, in which only the interaction of the α-th pair of particles is taken into account. All these operators act in the Hilbert space $L_2(R^6)$. A more detailed description of them may be found in the work of L. D. Faddeev (^{1}), the results of which are used essentially in the present note.

Suppose that the Fourier transforms $v_\alpha(k)$, $k \in R^3$, of the potentials of the pair interactions satisfy the conditions: 1) $v_\alpha(-k) = v_\alpha(k)$; 2) there exist continuous derivatives $D^\chi v_\alpha$ with respect to k of order $|\chi|$, $0 \leq |\chi| \leq 5$; 3) $|(D^\chi v_\alpha)(k)| \leq C(1 + |k|)^{-1-\theta}$, $\theta > 1/2$; 4) the energy operators h_α of the α-th pair, with the motion of its center of inertia separated out, have only a finite number of negative eigenvalues $-\chi^2_{\alpha,i}$, $-\chi^2_{\alpha,i} \in \mathfrak{S}_\alpha$; 5) the point $z = 0$ is not an exceptional point of certain special integral operators $a_\alpha(z)$ (see (^{1}), p. 26).

The connected part of the resolvent $R(z)$ is the operator

$$\mathfrak{R}(z) = R(z) - R_0(z) - \sum_\alpha (R_\alpha(z) - R_0(z)), \tag{1}$$

defined outside the spectrum Σ of the operator H.

Lemma 1. The operator

$$\mathfrak{R}_I(z) = \mathfrak{R}(z) - \mathfrak{R}(\bar{z}), \qquad z \in \Sigma, \tag{2}$$

is a nuclear operator.

Let $\Phi = \mathfrak{S}_1 \cup \mathfrak{S}_2 \cup \mathfrak{S}_3 \cup \Sigma_0$, where Σ_0 is the set of eigenvalues of the operator H.

Lemma 2. Let $\lambda \in \Phi$; then the limit

$$\Omega_\pm(\lambda) = \lim_{\varepsilon \to \pm 0} \operatorname{Sp} \mathfrak{R}_I(\lambda + i\varepsilon) \tag{3}$$

exists.

In ([2]), for the analogous expression $\omega(\lambda)$ in the case of an operator of type h (the energy of a pair), the trace formula

$$\omega_{\pm}(\lambda) = \frac{d}{d\lambda}\operatorname{Sp}\ln s(\lambda) = \operatorname{Sp} s^*(\lambda)\frac{ds(\lambda)}{d\lambda}, \qquad \lambda > 0, \tag{4}$$

was obtained, where $s(\lambda)$ is the scattering matrix for the operator h. A similar formula is proved below for $\Omega(\lambda)$.

The trace formula for $\Omega(\lambda)$ was studied within the framework of the formal perturbation theory of F. A. Berezin ([3]). It was assumed there that $\Phi = \emptyset$, so that the scattering was considered one-channel. In the present work the trace formula is derived without this restriction; moreover, in the case of one-channel scattering, our results differ somewhat in notation from F. A. Berezin' s formula, and we have not succeeded in verifying their identity.

2. Scattering matrix.

Consider the restriction of the operator H to the absolutely continuous subspace $\mathfrak{H}$. The usual direct-integral decomposition corresponding to it has the form

$$\mathfrak{H} \leftrightarrow \int_{-\chi^2}^{\infty} \oplus \mathfrak{H}(\lambda)\, d\lambda, \qquad -\chi^2 = \min(-\chi^2_{\alpha,i}). \tag{5}$$

The arrow $\leftrightarrow$ denotes unitary equivalence. For $\lambda < 0$, $\lambda \in \Phi$,

$$\mathfrak{H}(\lambda) = \sum_{-\chi^2_{\alpha,i}<\lambda} \oplus \mathfrak{H}_{\alpha,i}(\lambda), \qquad \mathfrak{H}_{\alpha,i}(\lambda) = L_2(S^2_\alpha, \rho_\alpha(\lambda + \chi^2_{\alpha,i})d\omega_\alpha). \tag{6}$$

Here $L_2(S^2_\alpha, \rho_\alpha(\lambda + \chi^2_{\alpha,i})d\omega_\alpha)$ is the space of functions on $S^2_\alpha = \{\omega = (2n_\alpha)^{1/2}p/|p| \ (p \in R^3)\}$, square-summable with respect to the measure $\rho_\alpha(\lambda + \chi^2_{\alpha,i})d\omega_\alpha$, $\lambda = p^2/2n_\alpha - \chi^2_{\alpha,i}$; $d\omega_\alpha$ is the surface element on S^2_α, and $\rho_\alpha(\lambda + \chi^2_{\alpha,i})d\omega_\alpha d\lambda = d^3p$. The constants denoted by the letters m, n, m_α, n_α are various "reduced masses" described in (1). For $\lambda > 0$, in (5) the direct sum (6) is supplemented by the term

$$\mathfrak{H}_0(\lambda) = L_2(S^5_0, \rho_0(\lambda)d\omega_0), \tag{7}$$

where

$S^5_0 = \{\omega = (k^2/2m + p^2/2n)^{-1/2}P \mid P = \{k, p\};\ k, p \in R^3\}; \qquad \lambda = k^2/2m + p^2/2n;$

$d\omega_0$ is the surface element on S^5_0, and $\rho_0(\lambda)d\omega_0 d\lambda = d^3k\, d^3p$.

The scattering operator $S : \mathfrak{H} \to \mathfrak{H}$ is given in the representation (5) by a family of unitary operators—the scattering matrix $S(\lambda) : \mathfrak{H}(\lambda) \to \mathfrak{H}(\lambda)$. The operators $S(\lambda)$ can be explicitly described by matrix kernels. In particular, for $\lambda < 0$ the kernels define the operator $S(\lambda)$ by the formula

$$g_{\alpha,i}(\omega_\alpha) = \sum_{\beta,k} \int_{S^2_\beta} S^{i,k}_{\alpha\beta}(\omega_\alpha, \omega'_\beta; \lambda)\, f_{\beta,k}(\omega'_\beta)\, d\omega'_\beta, \tag{8}$$

where $g = S(\lambda)f$, $g = \{g_{\alpha,i}(\omega_\alpha)\}$, $f = \{f_{\alpha,i}(\omega_\alpha)\}$. The kernels describing the action of $S(\lambda)$ on $f \in \mathfrak{H}_0(\lambda)$, $\lambda > 0$, are described analogously. The kernels defining $S(\lambda) - I$, I being the identity transformations of $\mathfrak{H}(\lambda)$, for $\lambda < 0$ are smooth functions of their arguments, while for $\lambda > 0$, generally speaking, they are singular generalized functions. The expression $dS(\lambda)/d\lambda$ will be characterized conventionally by the matrix kernels obtained by differentiating the kernels of the operator $S(\lambda)$ with respect to the variable λ.

3. Preparatory formulas.

With the aid of Hilbert' s identity the following can be proved.

Basic preparatory formula. If $z = \lambda + i\mu$, $z \in \Sigma$, then the relation holds

$$\operatorname{Sp} \mathfrak{R}_I(z) = \operatorname{Sp} \left\{ A^*(z) \frac{dA(z)}{d\lambda} - \sum_\alpha A^*_\alpha(z) \frac{dA_\alpha(z)}{d\lambda} \right\}, \tag{9}$$

where $A(z) = E - 2i\mu R_0(z) T(z) R_0(\bar{z})$ is a unitary operator, and $T(z)$ is related to the resolvent by the relation

$$R(z) = R_0(z) - R_0(z) T(z) R_0(z).$$

At the same time $A_\alpha(z)$ is analogously related to the operators $T_\alpha(z)$ and $R_\alpha(z)$.

We shall carry out the further transformations of this formula in the momentum representation (see (1)).

By integration by parts in (9), two relations are established. **Preparatory formula for the senior channel:**

$$\operatorname{Sp} \mathfrak{R}_I(z) = \operatorname{Sp} \left\{ \mathcal{P}[T(z)] - \sum_\alpha \mathcal{P}[T_\alpha(z)] \right\}. \tag{10}$$

where

$$\mathcal{F}[A(z)] = -2i\mu R_0(z) R_0(\bar{z}) \nabla_0 A(z) - (2i\mu)^2 R_0(z) R_0(\bar{z}) A(\bar{z}) R_0(z) R_0(\bar{z}) \nabla_0 A(z), \tag{11}$$

where the operator $\nabla_0 A(z)$ is given by the kernel

$$\nabla_0 A(z) \sim \rho_0^{-1}(P'^2)\left(\frac{\partial}{\partial\lambda}+\frac{\partial}{\partial\bar{P}^2}+\frac{\partial}{\partial P'^2}\right)\rho_0(P'^2)A(P,P';\lambda+i\mu), \tag{12}$$

where $A(P,P';z)$ is the kernel of the operator $A(z)$ in $L_2(R^6)$; differentiation with respect to P^2 is carried out for fixed $(P^2)^{-1/2}P=\omega_0$; the operations with respect to P' are defined analogously.

Preparatory formula for the junior channels:

$$\operatorname{Sp}\mathfrak{R}_1(z)=\operatorname{Sp}\left\{Q[T(z)]-\sum_{\alpha}Q[T_\alpha(z)]\right\}, \tag{13}$$

where

$$Q[T(z)]=-(2i\mu)^2\sum_{\alpha,\beta,\beta',\alpha'}R_0(z)R_0(\bar{z})M_{\alpha\beta}(z)R_0(z)R_0(\bar{z})\nabla_{\alpha\beta\beta'\alpha'}M_{\beta'\alpha'}(z)+$$

$$+2i\mu\left\{\sum_{\alpha\neq\alpha'}[M_{\alpha\alpha'}(z)-M_{\alpha\alpha'}(\bar{z})]\left[\frac{d}{d\lambda}R_0(z)R_0(\bar{z})\right]-\frac{d}{d\lambda}[R_0(z)R_0(\bar{z})T(z)]\right\}. \tag{14}$$

Here $M_{\alpha\beta}(z)$ $(\alpha,\beta=1,2,3)$ are the operators introduced in (1):

$$M_{\alpha\beta}(z)=\delta_{\alpha\beta}V_\alpha-V_\alpha R(z)V_\beta, \qquad V_\alpha=H_\alpha-H_0,$$

where

$$T(z)=\sum_{\alpha,\beta}M_{\alpha\beta}(z).$$

Further: the operator $\nabla_{\alpha\beta\beta'\alpha'}M_{\beta'\alpha'}(z)$ is given by the kernel

$$\left\{\frac{\partial}{\partial\lambda}+\delta_{\beta\beta'}2n_\beta\frac{\partial}{\partial p'^2_\beta}+\right.$$

$$\left.+\delta_{\alpha\alpha'}\rho_\alpha^{-1}\left(\frac{p'^2_\alpha}{2n_\alpha}\right)\cdot 2n_\alpha\frac{\partial}{\partial p^2_\alpha}\rho_\alpha\left(\frac{p'^2_\alpha}{2n_\alpha}\right)\right\}M_{\beta'\alpha'}(P,P';\lambda+i\mu), \tag{15}$$

where M with arguments is the kernel of $M(z)$. Differentiation with respect to p_α^2 is carried out for fixed

$$\omega_\alpha = (2n_\alpha)^{1/2} \frac{p_\alpha}{|p_\alpha|}$$

and k_α. The pairs of variables $\{k_\alpha, p_\alpha\}$ used to describe P are introduced in the same way as in (1).

4. Trace formula. The result of passing to the limit $\mu \to 0$ in formulas (10) and (13) is determined by the singularities of the kernel of the operator $T(z)$ (see (1)).

Trace formula for the junior channels. Let $\lambda < 0$, $\lambda \in \Phi$; then, as $\mu \to 0$, in (13) only the first terms in $Q[T(z)]$, corresponding to $\alpha = \alpha'$, $\beta = \beta'$, are retained.

Theorem 1. *In the limit we obtain*

$$\Omega_\pm(\lambda) = \pm \operatorname{Sp} \mathfrak{S}_{(\lambda)} S^*(\lambda)\, dS(\lambda)/d\lambda. \tag{16}$$

Here $dS(\lambda)/d\lambda$ is given by differentiated kernels, as described in Sec. 2. The trace on the right-hand side of (16) is understood as the trace of the kernel.

Trace formula for the senior channel. Let $\lambda > 0$, $\lambda \in \Sigma_0$. For simplicity we shall assume the scattering to be single-channel, i.e.

$$\mathfrak{S}_1 \cup \mathfrak{S}_2 \cup \mathfrak{S}_3 = \emptyset.$$

Put

$$\dot{T}(z) = \sum_\alpha T_\alpha(z) - \sum_{\alpha \neq \beta} T_\alpha(z) R_0(z) T_\beta(z).$$

$\dot{T}(z)$ contains the senior singularities of the kernel $T(z)$. The operator $S(\lambda)$ is expressed explicitly in terms of $T(z)$. We denote by $\dot{S}(\lambda)$ the contribution to $S(\lambda)$ from $\dot{T}(z)$.

Theorem 2. Passing to the limit in formula (10) gives

$$\Omega_\pm(\lambda) = \pm \operatorname{Sp}_{\mathcal{C}(\lambda)} \left[S^*(\lambda) \frac{dS(\lambda)}{d\lambda} - \mathring{S}^*(\lambda) \frac{d\mathring{S}(\lambda)}{d\lambda} \right] \pm \Delta(\lambda), \tag{17}$$

where

$$\Delta(\lambda) = \lim_{\mu \downarrow 0} \operatorname{Sp} \left\{ \mathfrak{P}\,[\dot{T}(z)] - \sum_a \mathfrak{P}\,[T_a(z)] \right\}. \tag{18}$$

Here the same explanations must be made as in Theorem 1. It is clear that $\Delta(\lambda)$ is a polynomial in the "pair" operators $T_a(z)$, which enter into $\Delta(\lambda)$ at most to the fourth degree. They are explicitly expressed in terms of the resolvents of the operators h_a.

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Note: Figure translations are in progress. See original paper for figures.

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