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CORRESPONDING TO
A CERTAIN PART OF
THE SET OF
EIGENVALUES OF A
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OPERATOR OF
SECOND ORDER**

MATHEMATICS

1969

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Abstract

Full Text

UDC 517.94

MATHEMATICS

O. A. AKHVEROVA, M. G. DZHAVADOV

**ON AN EXPANSION IN EIGENFUNCTIONS
CORRESPONDING TO A CERTAIN PART
OF THE SET OF EIGENVALUES OF A NON-
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(Presented by Academician I. G. Petrovskii on 10 II 1969)

In the fundamental work ⁽¹⁾ M. V. Keldysh, along with other facts, proved the multiple completeness of all eigenfunctions and associated functions of a non-self-adjoint differential operator. In the works ^(5, 6) the multiple completeness of the eigenfunctions of a non-self-adjoint differential operator corresponding to a certain part of the set of eigenvalues was proved, and in the work ⁽⁷⁾ similar questions were investigated for abstract operators.

In the present note it is proved that a function $\varphi(t)$, having derivatives up to the second order and satisfying zero boundary conditions, is expanded in a series in the eigenfunctions of a non-self-adjoint differential operator of second order corresponding to negative eigenvalues.

Consider the problem:

$$y'' + 2A\lambda y' + B\lambda^2 y = 0; \tag{1}$$

$$y(0) = 0, \quad y(1) = 0, \tag{2}$$

where A and B are constants such that $A^2 - B < 0$; λ is a parameter.

By direct computation one can verify that the eigenvalues of problem (1) and (2) are

$$\lambda_k = ak, \quad k = \pm 1, \pm 2, \dots, \quad a = \pi/\sqrt{B - A^2} > 0.$$

Denote by $\{y_k^-\}$ the set of all eigenfunctions corresponding to negative eigenvalues. Before proceeding to the proof of the basis property of this set of functions, consider an auxiliary problem.

Let $\Omega = \{(t, \tau), 0 \leq t \leq 1, 0 \leq \tau < \infty\}$ be a half-strip. In Ω we seek solutions of the problem, decreasing at infinity,

$$\partial^2 u / \partial t^2 + 2A \partial^2 u / \partial t \partial \tau + B \partial^2 u / \partial \tau^2 = 0; \quad (3)$$

$$u|_{t=0} = 0, \quad u|_{t=1} = 0, \quad u|_{\tau=0} = \varphi(t), \quad (4)$$

where $A^2 - B < 0$, and $\varphi(t)$ is a given smooth function.

By functional methods it is proved that problem (3), (4) has a unique solution belonging to the space $\dot{W}_2^1(\Omega)$.

In what follows we shall need to study the behavior of the derivatives of the solutions of problem (3), (4) for $\tau = 0$. From the results of the work ⁽⁴⁾ it follows that on the smooth part of the boundary the solution of problem (3) and (4) will be a smooth function; the behavior of the higher derivatives of this function can deteriorate only near the corner points.

We shall study the nature of the solution and of its first derivative near the corner point $(0, 0)$. For this, first, by the obvious substitution of the unknown function

we reduce problem (3), (4) to a problem with homogeneous boundary conditions, then bring the resulting equation to canonical form and write it in polar coordinates

$$\frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial w}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 w}{\partial \alpha^2} = f(\rho, \alpha); \quad (5)$$

$$w|_{\alpha=0} = 0, \quad w|_{\alpha=\theta} = 0, \quad (6)$$

where θ is the angle into which a right angle is transformed when equation (3) is brought to canonical form.

Applying the transform

$$\tilde{w}(s, \alpha) = \int_0^\infty \rho^{s-1} w(\rho, \alpha) d\rho \quad (7)$$

to problem (5), (6), we obtain

$$\frac{\partial^2 \tilde{w}}{\partial \alpha^2} + s^2 \tilde{w} = \tilde{f}(s + 2, \alpha); \quad (8)$$

$$\tilde{w}|_{\alpha=0} = 0, \quad \tilde{w}|_{\alpha=\theta} = 0. \quad (9)$$

Obviously, the solution of problem (8), (9) is written in the form

$$\tilde{w}(s, \alpha) = \int_0^\theta H(s, \alpha, \beta) \tilde{f}(s + 2, \beta) d\beta,$$

where

$$H(s, \alpha, \beta) = \begin{cases} -\frac{\sin s\beta}{\sin s\theta} \frac{\sin s(\theta - \alpha)}{s}, & \text{for } \beta < \alpha, \\ -\frac{\sin s\alpha}{\sin s\theta} \frac{\sin s(\theta - \beta)}{s}, & \text{for } \beta > \alpha. \end{cases}$$

Then

$$w(\rho, \alpha) = \int_0^\infty r dr \int_0^\theta G(\rho, \alpha; r, \beta) f(r, \beta) d\beta,$$

where

$$G(\rho, \alpha; r, \beta) = \begin{cases} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{r}{\rho}\right)^s \frac{\sin s\beta}{\sin s\theta} \frac{\sin s(\theta - \alpha)}{s} ds, & \text{for } \beta < \alpha, \\ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \left(\frac{r}{\rho}\right)^s \frac{\sin s\alpha}{\sin s\theta} \frac{\sin s(\theta - \beta)}{s} ds, & \text{for } \beta > \alpha. \end{cases} \quad (10)$$

Computing the residues at the poles of the integrand in (10), it is easy to verify that the function $w(\rho, \alpha)$ and its first derivative with respect to ρ are continuous as $\rho \rightarrow 0$. Consequently, the function $u(t, \tau)$ and its first derivative with respect to τ are continuous at $\tau = 0$.

We proceed to the proof of the main fact, i.e., that the set $\{y_k^-\}$ forms a basis. Applying the transform

$$y(\lambda, t) = \int_0^\infty e^{-\lambda\tau} u(t, \tau) d\tau \quad (11)$$

to problem (3), (4), we obtain

$$y'' + 2A\lambda y' + B\lambda^2 y = F(\lambda, t); \quad (12)$$

$$y(0) = 0, \quad y(1) = 0, \quad (13)$$

where

$$F(\lambda, t) = 2A\varphi'(t) + B\psi(t) + B\lambda\varphi(t), \quad \psi(t) = \partial u / \partial t|_{\tau=0}.$$

The solution of problem (12), (13) has the form

$$y(\lambda, t) = \int_0^1 g(\lambda, t, \xi) F(\lambda, \xi) d\xi,$$

where $g(\lambda, t, \xi)$ is the Green's function of problem (12), (13). It is known⁽³⁾ that it has the representation

$$g(\lambda, t, \xi) = \sum_{n=1}^{\infty} \frac{\Phi_n(t)\Psi_n(\xi)}{\lambda - \lambda_n},$$

where $\Phi_n(t)$ are the eigenfunctions of problem (1), (2), and $\Psi_n(t)$ are the eigenfunctions of the corresponding adjoint problem.

Passing to the transform inverse to (11), we obtain

$$u(t, \tau) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{\lambda\tau} \sum_{n=1}^{\infty} c_n \frac{\Phi_n(t)}{\lambda - \lambda_n} d\lambda, \quad (14)$$

where

$$c_n = \int_0^1 \Psi_n(\xi) F(\lambda, \xi) d\xi.$$

It is known⁽³⁾ that in the λ -plane one can construct a system of contours Γ_N possessing the following properties: 1) none of the eigenvalues lies on Γ_N ; 2) Γ_N is entirely contained inside Γ_{N+1} ; 3) the shortest distance R_N from the origin to Γ_N increases without bound as N increases; 4) on the contours Γ_N the Green's function satisfies the estimate

$$|g(\lambda, t, \xi)| \leq \text{const}/|\lambda|.$$

Replacing the limits of integration from $c - i\infty$ to $c + i\infty$ in expression (14) by Γ_N , taking into account the analyticity of the function $y(\lambda, t)$ in the right half-plane, and computing the residues at the poles of the Green's function, we obtain

$$u(t, \tau) = \sum_{n=1}^{\infty} c_n e^{-an\tau} \Phi_n(t). \quad (15)$$

By repeating the argument of paper (6), it is proved that the function $u(t, \tau)$, defined by equality (15), is a solution of problem (3), (4).

We now prove that the series converges

$$\sum_{n=1}^{\infty} c_n \Phi_n(t).$$

Indeed,

$$\sum_{n=1}^N c_n \Phi_n(t) = \frac{1}{2\pi i} \int_{\Gamma_N} d\lambda \int_0^1 g(\lambda, t, \xi) F(\lambda, \xi) d\xi. \quad (16)$$

We shall show that the limit of (16) exists as $N \rightarrow \infty$.

It is known that the Green's function $g(\lambda, t, \xi)$, with respect to ξ , is the solution of the problem:

$$\partial^2 g / \partial \xi^2 - 2A\lambda \partial g / \partial \xi + B\lambda^2 g = \delta(t - \xi),$$

$$g|_{\xi=0} = 0, \quad g|_{\xi=1} = 0.$$

Hence

$$g = \frac{1}{B\lambda^2} \left\{ \delta(t - \xi) + 2A\lambda \frac{\partial g}{\partial \xi} - \frac{\partial^2 g}{\partial \xi^2} \right\}. \quad (17)$$

Substituting the expression for $g(\lambda, t, \xi)$ from (17) into the right-hand side of (16), we

we obtain

$$\int_{\Gamma_N} d\lambda \int_0^1 g(\lambda, t, \xi) F(\lambda, \xi) d\xi =$$

$$= \int_{\Gamma_N} \frac{d\lambda}{B\lambda^2} \int_0^1 \left\{ \delta(t - \xi) + 2A\lambda \frac{\partial g}{\partial \xi} - \frac{\partial^2 g}{\partial \xi^2} \right\} \{2A\varphi'(\xi) + B\psi(\xi) + B\lambda\varphi(\xi)\} d\xi. \quad (18)$$

Multiplying the expressions under the integral sign in (18), we then examine each term separately. Consider the integral

$$\int_{\Gamma_N} \frac{d\lambda}{\lambda^2} \int_0^1 \frac{\partial^2 g(\lambda, t, \xi)}{\partial \xi^2} \psi(\xi) d\xi. \quad (19)$$

Using the explicit expression for the function $g(\lambda, t, \xi)$ and the continuity of the function $\psi(\xi)$, it is easy to prove that the integral (19) exists in the sense of the principal value.

The existence of the remaining integrals in (18), under the assumption that the function $\varphi(t)$ is twice continuously differentiable, is obvious.

Thus, we have shown that as $N \rightarrow \infty$ there exists a limit of the expression

$$\sum_{n=1}^N c_n \Phi_n(t).$$

Taking into account that the series

$$\sum_{n=1}^{\infty} c_n e^{-an\tau} \Phi_n(t)$$

converges for $\tau > 0$ and, as $\tau \rightarrow 0$, tends to the function $\varphi(t)$, we assert that

$$\varphi(t) = \sum_{n=1}^{\infty} c_n \Phi_n(t).$$

Thus, the following has been proved.

Theorem. *If the function $\varphi(t)$ has continuous derivatives up to and including the second order, and moreover $\varphi(0) = \varphi(1) = 0$, then it can be expanded in a series in the eigenfunctions of problem (1), (2) corresponding to the negative eigenvalues, i.e., in the eigenfunctions $\{y_k\}$.*

Remark. Obviously, the proposed approach carries over without any change to a nonself-adjoint differential operator of arbitrary order.

Institute of Mathematics and Mechanics
Academy of Sciences of the Azerbaijan SSR
Baku

Received
18 I 1969

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