THE HOROCYCLIC POCKET

MATHEMATICS

1969

SovietRxiv

Abstract

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UDC 513.821.838

MATHEMATICS

D. A. DE-SHILLER

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(Presented by Academician L. S. Pontryagin, 23 IX 1968)

In the report "The geometry of proximity of Riemannian manifolds" at a meeting of the Moscow Mathematical Society on May 12, 1953 (see ([1])), V. A. Efremovich put forward the supposition that the so-called horocyclic pocket, i.e., the universal covering of the Beltrami pseudosphere, is not equimorphic to any surface of revolution. Here this assertion is proved.

The horocyclic pocket can also be defined in another way: it is the surface obtained by identifying corresponding boundary points of two inner domains bounded in the Lobachevsky plane H^2 by nonintersecting horocycles, the corresponding points being those which are carried into one another by some motion taking one horocycle into the other. Each of these domains will be called a sheet. In what follows, when speaking of a horocycle, we shall mean precisely that horocycle along which the edges of the two sheets have been identified. Any two points of the horocycle are the endpoints of two rectilinear segments lying on different sheets. Together they form a closed line—a two-gon. The part of the pocket bounded by such a two-gon will be called a two-sheeted segment of the pocket, or simply a segment. The part of the horocycle contained inside the segment will be called the arc of the segment, and the point lying at its midpoint—the center of the segment.

Theorem. *The horocyclic pocket is not equimorphic to any surface of revolution.*

We first prove two auxiliary assertions $\mathcal{A}$ and $\mathcal{B}$.

$\mathcal{A}$. *If there exists an equimorphism f mapping the horocyclic pocket K onto some surface of revolution R, then there exists a set $\{\varphi\}$ of equimorphisms mapping K onto itself which satisfies the following conditions:*

I. *There exists a function $\lambda(\varepsilon)$, defined for all positive ε and taking only positive values, such that if a_1 and a_2 are points of K and the distance $\rho(a_1, a_2) \geq \varepsilon$, then for any equimorphism $\varphi_\alpha \in \{\varphi\}$*

$$\frac{1}{\lambda(\varepsilon)}\, \rho(a_1, a_2) < \rho(\varphi_\alpha(a_1), \varphi_\alpha(a_2)) < \lambda(\varepsilon)\, \rho(a_1, a_2). \tag{1}$$

II. *For any positive number d there exists an equimorphism $\varphi_\alpha \in \{\varphi\}$ taking the horocycle into a curve containing points whose distance from the horocycle is greater than d.*

Without loss of generality, suppose that under the equimorphism f some point o belonging to the horocycle goes to the center o' of revolution of the surface R. Denote by g_α the rotation of R through an angle $\alpha \leq 2\pi$ about o' in a prescribed direction. Denote by $\{\varphi\}$ the set of equimorphisms $\varphi_\alpha = f^{-1} g_\alpha f$. By the properties of equimorphisms of geodesic spaces ([2], [3]), for the equimorphisms f and f^{-1} there exist functions $\lambda_1(\varepsilon)$ and $\lambda_2(\varepsilon)$, defined for all positive ε and taking only positive values, such that if a_1 and a_2 are points of K and $\rho(a_1, a_2) \geq \varepsilon$, then

$$\frac{1}{\lambda_1(\varepsilon)}\, \rho(a_1, a_2) < \rho(f(a_1), f(a_2)) < \lambda_1(\varepsilon)\, \rho(a_1, a_2), \tag{2}$$

and if b_1 and b_2 are points of R and $\rho(b_1, b_2) \geq \varepsilon$, then

$$\frac{1}{\lambda_2(\varepsilon)}\, \rho(b_1, b_2) < \rho(f^{-1}(b_1), f^{-1}(b_2)) < \lambda(\varepsilon)\, \rho(b_1, b_2). \tag{2'}$$

Put $\lambda(\varepsilon) = \lambda_2(\varepsilon/\lambda_1(\varepsilon))\lambda_1(\varepsilon)$. Taking into account that every equimorphism $\varphi_\alpha \in \{\varphi\}$ is equal to $f^{-1} g_\alpha f$, where g_α is a motion, and taking into account the inequalities (2) and (2′), it is easy to compute that, for $a_1, a_2 \in K$, $\rho(a_1, a_2) \geq \varepsilon$, every equimorphism $\varphi_\alpha \in \{\varphi\}$ satisfies the inequality (1). Thus, for the set $\{\varphi\}$, condition I is fulfilled. Let us prove that condition II is also fulfilled. Let A be the axis of symmetry of one of the leaves and $o \in A$. Let B be one of the branches of the horocycle issuing from o, with $o \notin B$. Let x be an arbitrary point of B. Denote by Γ_x the circle on R with center at o', passing through the point $f(x)$, and let $y(x)$ be one of the points of intersection $f(A) \cap \Gamma_x$. (Obviously, $f(A) \cap \Gamma_x \neq \emptyset$.) Let $\alpha(x) < 2\pi$ be an angle such that $g_{\alpha(x)} f(x) = y(x)$. Then $y = \varphi_{\alpha(x)}(x) = f^{-1} g_\alpha f(x) \in A$. Since every equimorphism $\varphi_\alpha \in \{\varphi\}$ leaves the point o fixed, it follows from inequality (1) that, when the point $x \in B$ recedes without bound from o, the point $\varphi_{\alpha(x)}(x) \in A$ also recedes without bound from o, and hence from the horocycle.

$\mathcal{B}$. Let Q be a closed domain in H^2, bounded by a curve ∂Q that is homeomorphic to a circle; let the ε-capacity* of ∂Q be equal to $n+1$, $n \geq 1$. Then the 4ε-capacity of Q does not exceed n^2.

In the proof we shall use only the following property of H^2: for any point $o \in H^2$ and any two rays A and B issuing from it, $\rho(y, B)$ does not decrease when the point $y \in A$ recedes from o. Since the ε-capacity of ∂Q is equal to $n+1$, there exist $n+1$ points $x_0, x_1, \dots, x_n$ forming an ε-net on ∂Q. Join the point x_0 to the points $x_1, \dots, x_n$ by segments. In view of the indicated property of H^2, every point $z \in Q$ is at a distance from one of the constructed segments of less than ε, since this is true for every point $z^* \in \partial Q$; moreover, obviously, $\rho(x_0, x_i) < 2n\varepsilon$

for $i = 1, \dots, n$. Taking this into account, it is easy to see that the domain Q can be covered by n^2 disks of radius $r < 2\varepsilon$ (for example with centers on the constructed segments), and hence $\mathcal{B}$ follows immediately.

We proceed to the proof of our theorem by contradiction. We retain the notation $\{\varphi\}$ and $\lambda(\varepsilon)$ from $\mathcal{A}$. It is clear that the 1-capacity m of any curve of length l satisfies the inequality

$$m - 1 \leq l. \tag{3}$$

Next, there exists a sufficiently small positive δ such that the $4(\lambda(1))^2$-capacity N of any horocycle arc of length L satisfies the inequality

$$N \geq \delta L. \tag{4}$$

Let now L denote the length of the arc and $2a$ the length of the boundary ∂S of such a segment S that

$$(\lambda(1))^2 \leq a < \left[\frac{1}{2}\right] \delta L. \tag{5}$$

Such a segment exists, since the length of a horocycle arc is equal to twice the hyperbolic sine of half the length of the chord subtending it. Let the 1-capacity of ∂S be equal to $n+1$ and the $4(\lambda(1))^2$-capacity of the arc of the segment S be equal to N. From inequalities (3), (4), and (5) it follows that

$$n^2 < N. \tag{6}$$

Let o be the center of S and $\rho(o, \partial S) = d$. By virtue of condition II from $\mathcal{A}$, there exists an equimorphism $\varphi_\alpha \in \{\varphi\}$ which sends one of the points of the horocycle to a point o' whose distance from the horocycle is greater than $\lambda(d)a$. Without loss of generality

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* The ε-capacity n_M^ε of a bounded set M is the maximal number of points that can be placed in M so that the distances between any two of them are not less than ε (see ([1]), p. 189 or ([3]), p. 76).

In general, one may suppose that $o' = \varphi_a(o)$. Then, since the diameter S is equal to a, the image S' of the segment S under the equimorphism φ_a lies entirely on one sheet, that is, on H^2. Denote by P and p, respectively, the $4\lambda(1)$-capacity of S' and the $\lambda(1)$-capacity of the boundary $\partial S'$. From consideration of the equimorphisms φ_a^{-1} and φ_a it follows easily that $p \geq n+1$ and $P \geq N$. From inequality (5) it follows that $2 \leq p$. The latter inequalities, considered together with inequality (6), contradict $\mathcal{B}$. The contradiction obtained proves the theorem.

I express my gratitude to V. A. Efremovich for his attention to this work and for his help.

Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

Received
5 VIII 1968

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[1] V. A. Efremovich, UMN, 8, 5, 189 (1953). [2] V. A. Efremovich, UMN, 4, 2, 179 (1949). [3] V. A. Efremovich, Scientific Notes of the Ivanovo Pedagogical Institute, 31, 74 (1963).

Note: Figure translations are in progress. See original paper for figures.