ON A GENERALIZATION OF THE INTEGRAL FORMULAS OF CAUCHY, SCHWARZ, AND POISSON

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Abstract

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I. I. Bavrin

ON A GENERALIZATION OF THE INTEGRAL FORMULAS OF CAUCHY, SCHWARZ, AND POISSON

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M. M. Dzhrbashyan ([1]) constructed a generalized operator $L^{(\omega)}$ of Riemann-Liouville type, by means of which he established fundamentally new analogues of the classical formulas of Cauchy, Schwarz, and Poisson*–generalized Cauchy, Schwarz, and Poisson formulas associated with a given function $\omega(x) \in \Omega$**. In the present note this generalized operator is used to establish (Theorems 1, 2) generalized Cauchy, Schwarz, and Poisson formulas associated with a given system of functions $\omega_j(x) \in \Omega$ $(j = 1, 2, \dots, m)$.

Let the functions $\omega_j(x) \in \Omega$ $(j = 1, 2, \dots, m)$. Further, let $p_j(0) = 1$,

$$p_j(r) = r \int_r^1 \frac{\omega_j(x)}{x^2}\, dx \quad (r \in (0,1]), \qquad \Delta_0^{(j)} = 1, \quad \Delta_k^{(j)} = -(k+1) \int_0^1 r^k\, dp_j(r) =$$

$$= k \int_0^1 r^{k-1} \omega_j(r)\, dr \quad (j = 1, 2, \dots, m), \quad k = 1, 2, \dots^{***}.$$

We introduce for consideration the power series

$$C(z; \omega_1, \dots, \omega_m) = \sum_{k=0}^{\infty} \frac{z^k}{\Delta_k^{(1)} \cdots \Delta_k^{(m)}}. \tag{1}$$

It is easy to see that the radius of convergence of this series is equal to one. Thus the function $C(z; \omega_1, \dots, \omega_m)$ is holomorphic in the disk $|z| < 1$. Along with this function we also introduce the function

$$S(z; \omega_1, \dots, \omega_m) = 2C(z; \omega_1, \dots, \omega_m) - C(0; \omega_1, \dots, \omega_m) =$$

$$= 1 + 2\sum_{k=1}^{\infty} \frac{z^k}{\Delta_k^{(1)} \cdots \Delta_k^{(m)}}, \tag{2}$$

noting that $C(0; \omega_1, \dots, \omega_m) = 1/\Delta_0^{(1)} \cdots \Delta_0^{(m)} = 1$.

* For other generalizations of the Cauchy, Schwarz, and Poisson formulas, see, for example, ([2–6]).

** It is said (see ([1]), p. 1078) that a function $\omega(x) \in \Omega$ if it is nonnegative and continuous on $[0, 1)$, with $\omega(0) = 1$,

$$\int_0^1 \omega(x)\, dx < +\infty$$

and for every r $(0 \le r < 1)$

$$\int_r^1 \omega(x)\, dx > 0.$$

*** In ([1]) the function $p(0) = 1$,

$$p(r) = r\int_r^1 \frac{\omega(x)}{x^2}\, dx \quad (\omega(x) \in \Omega), \quad r \in (0, 1],$$

and the sequence of numbers

$$\Delta_k = -(k+1)\int_0^1 r^k\, dp(r) \quad (k = 0, 1, 2, \dots)$$

were introduced, and it was shown that all the numbers Δ_k $(k = 0, 1, 2, \dots)$ are positive, with $\Delta_0 = 1$,

$$\Delta_k = k\int_0^1 \omega(x) x^{k-1}\, dx \quad (k = 1, 2, \dots).$$

Theorem 1. Let the function

$$f\left(re^{i\varphi}\right) = \sum_{k=0}^{\infty} a_k \left(re^{i\varphi}\right)^k$$

be holomorphic in the disk $|z| < R$. Then the function

$$L^{(\omega_m)}\left[L^{(\omega_{m-1})} \dots \left[L^{(\omega_1)}\left[f\left(re^{i\varphi}\right)\right]\right] \dots\right] \equiv$$

$$\equiv L^{(\omega_1,\dots,\omega_m)}\left[f\left(re^{i\varphi}\right)\right] \equiv f_{(\omega_1,\dots,\omega_m)}\left(re^{i\varphi}\right) = \sum_{k=0}^{\infty} \Delta_k^{(1)} \dots \Delta_k^{(m)} a_k \left(re^{i\varphi}\right)^k \qquad (3)$$

is holomorphic in the same disk $|z| < R$, and for any ρ $(0 < \rho < R)$ the integral formulas

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} C\left(e^{-i\theta}\frac{z}{\rho}; \omega_1, \dots, \omega_m\right) f_{(\omega_1,\dots,\omega_m)}\left(\rho e^{i\theta}\right) d\theta \quad (|z| < \rho),$$

$$f(z) = i \operatorname{Im} f(0) + \frac{1}{2\pi} \int_0^{2\pi} S\left(e^{-i\theta}\frac{z}{\rho}; \omega_1, \dots, \omega_m\right) \times$$

$$\times \operatorname{Re} f_{(\omega_1,\dots,\omega_m)}\left(\rho e^{i\theta}\right) d\theta \quad (|z| < \rho).$$

In the course of the proof, the Cauchy-Hadamard formula is essentially used, as well as the formulas

$$L^{(\omega_1,\dots,\omega_m)}[r^k] = \Delta_k^{(1)} \dots \Delta_k^{(m)} r^k \quad (k = 0, 1, 2, \dots)$$

and the expansions (1), (2), and (3).

Let us introduce into consideration the function

$$P(\theta, r; \omega_1, \dots, \omega_m) = \operatorname{Re} S\left(re^{i\theta}; \omega_1, \dots, \omega_m\right) = 1 + 2\sum_{k=1}^{\infty} \frac{r^k \cos k\theta}{\Delta_k^{(1)} \dots \Delta_k^{(m)}},$$

harmonic in the unit disk $0 \le r < 1,\ 0 \le \theta \le 2\pi$.

From Theorem 1 it follows easily:

Theorem 2. Let $u(z)$ be a harmonic function in the disk $|z| < R$. Then the function

$$u_{(\omega_1,\dots,\omega_m)}\left(re^{i\varphi}\right) = L^{(\omega_1,\dots,\omega_m)}\left[u\left(re^{i\varphi}\right)\right]$$

will be harmonic in the same disk $|z| < R$, and for any ρ $(0 < \rho < R)$ the integral formula

$$u\left(re^{i\varphi}\right) = \frac{1}{2\pi}\int_0^{2\pi} P\left(\varphi-\theta, \frac{r}{\rho}; \omega_1, \dots, \omega_m\right) u_{(\omega_1,\dots,\omega_m)}\left(\rho e^{i\theta}\right) d\theta$$

holds,

$$(0 \le r < \rho, \qquad 0 \le \varphi \le 2\pi).$$

Moscow Regional Pedagogical Institute
named after N. K. Krupskaya

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Note: Figure translations are in progress. See original paper for figures.