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Abstract

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THE GIBBS ENSEMBLE AND CHAINS OF EQUATIONS FOR CORRELATION FUNCTIONS OF WEAKLY RELATIVISTIC SYSTEMS

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In the present work we consider systems of classical (nonquantum) statistical mechanics for whose study it is sufficient to take relativistic effects into account only in the first approximation, i.e., one may neglect quantities of order $(v/c)^3$, $(v/c)^4$, ..., where v is the velocity of a particle belonging to the system, and c is the speed of light. Such systems will be called weakly relativistic. In order to obtain chains of recurrent equations of N. N. Bogolyubov determining the evolution of the correlation functions F_s, introduced in the monograph ([1]), one should start from the Liouville equation for the distribution function in Gibbs phase space. In relativistic theory, the states of particles are specified by giving the set of their coordinates q, relativistically invariant momenta, and proper times $(\tau_1, \tau_2, ...)$. In the work of N. A. Chernikov ([2]), a seven-dimensional relativistic phase space of states of one particle (μ-space) was constructed. However, a generalization of N. A. Chernikov' s method for constructing the Gibbs phase space turns out to be impossible, since the state of the system can be specified only by indicating N proper times, where N is the number of particles in the system, whereas what is of practical interest is a description of the system referred to laboratory time t. Synchronization of times, if it is possible, leads to the result that the momenta $p = dq/dt$ become noninvariant with respect to the Lorentz transformation group. The indicated difficulty is not the only one —a more detailed analysis of the situation is given in ([3]).

Phase space. Let $U(q)$ be the value of the gravitational-field potential at the point q. Then, with accuracy up to quantities of order c^{-3}, the metric tensor in the four-dimensional space of events has the form

$$g_{00} = 1 - 2Uc^{-2}; \qquad g_{jk} = -(1 + 2Uc^{-2}); \tag{1}$$

$$g_{0j} = g_{j0} = 0; \qquad (jk) = 1, 2, 3, 4.$$

Since $g_{0j} = g_{j0} = 0$, synchronization of the times $(\tau_1, \tau_2, \ldots, \tau_N)$ with the time t is possible. We shall assume it to have been carried out. We define the momentum of the j-th particle as in nonrelativistic theory: $p_j = m(dq_j/dt)$, where m is the rest mass of the particle. Then Lobachevsky geometry ([4]) takes place in momentum space. Let us note, incidentally, that the geometric properties of the space of relativistically invariant momenta would turn out to be much more complicated. The metric tensor of momentum space f_{jk} is determined by the formulas

$$f_{jk} = \frac{p_{jp}k}{(1 - p^2/m^2c^2)} \qquad (j \neq k), \tag{2}$$

$$f_{jj} = \frac{1}{(1 - p^2/m^2c^2)} \left(1 - \frac{p_j^2}{m^2c^2 - p^2}\right); \qquad j, k = 1, 2, 3.$$

By the general rules (5), it is not difficult to find the volume elements dV_q and dV_p in coordinate and momentum spaces, respectively:

$$dV_q = \frac{dx\,dy\,dz}{(1 + 2Uc^{-2})^2} \cong \frac{dx\,dy\,dz}{1 + 4U_{ec}^{-2}},$$

$$dV_p = \frac{dp_1\,dp_2\,dp_3}{(1 - p^2/m^2c^2)^2} \cong \frac{dp_1\,dp_2\,dp_3}{1 - 2p^2/m^2c^2}. \tag{3}$$

Here U_e is the potential of the gravitational field obtained in the "Newtonian" nonrelativistic theory in Euclidean space.

The dimensions of momentum space are determined by the inequalities

$$0 \leq p^2 \leq m^2c^2. \tag{4}$$

The phase space μ_j of the states of the j-th particle is obtained as the direct product of three-dimensional coordinate space with metric (1) and three-dimensional momentum space with metric (2). Its volume element is

$$d\gamma_j = dV_{q_j} dV_{p_j} \cong \frac{dx_j\,dy_j\,dz_j\,dp_{1j}\,dp_{2j}\,dp_{3j}}{(1 + 4U_{ec}^{-2})(1 - 2p_j^2/m^2c^2)}. \tag{5}$$

It follows from (4) and (5) that the volume of the space μ is infinite.

We shall define the Gibbs ensemble as the set of all possible systems that are macroscopically equivalent but pairwise different microscopically at $t = 0$. The dynamical equations specifying the motions of the particles of the ensemble will be written below.

The Gibbs phase space will be the direct product of the spaces μ_j:

$$\Gamma = \mu_1 \times \mu_2 \times ... \times \mu_N.$$

Its volume element is

$$d\Gamma = d\gamma_1 d\gamma_2 \dots d\gamma_N. \tag{6}$$

Dynamical equations. Starting from the Newton–Hertz equation

$$\frac{d}{dt}\frac{mv}{\sqrt{1 - v^2/c^2}} = \mathbf{F}, \tag{7}$$

where $\mathbf{F} = -\nabla_q W$, W is the total potential at the point where the particle is located, we shall write the dynamical equations determining the motion of particles in μ-space and the motion of systems in Γ-space. Let $\mathbf{F}_e$ denote the value of the force acting on the particle in Euclidean space in the nonrelativistic formulation of the problem. Expressing $\mathbf{F}$ through U_e and $\mathbf{F}_e$, and transferring from the left-hand side of (7) to the right-hand side all quantities except $d(m\mathbf{v})/dt = \dot{\mathbf{p}}$, and then discarding terms of higher order of smallness than $(v/c)^2$, we obtain the equation

$$\dot{\mathbf{p}} = [1 + 4U_{ec}^{-2} - {}^3/{}_2 v^2 c^{-2}]\mathbf{F}_e + o(c^{-3}). \tag{8}$$

Introducing the function

$$H = \sum_{(1 \le j \le N)} \left[\frac{p_j^2}{2m} + \left(1 + \frac{2U_e(q_j)}{c^2}\right) W(q_j) \right] \tag{9}$$

and the generalized force

$$Q_j = -\frac{3}{2}\frac{p_j^2}{m^2c^2}\mathbf{F}_e \tag{10}$$

and taking (8) into account, we arrive at the system of quasicanonical equations:

$$\dot{\mathbf{p}}_j = -\partial H/\partial q_j + Q_j, \qquad \dot{\mathbf{q}}_j = \partial H/\partial p_j. \tag{11}$$

Remark. If $U = 0$ everywhere, i.e. the gravitational fields are negligibly small, then instead of (9) one should write

$$H = \sum_{(j)} (p_j^2/2m) + \sum_{(j,k)} \Phi(|q_j - q_k|), \tag{12}$$

where $\Phi(|q_j + q_k|)$ is the interaction potential of the j-th and k-th particles.

Liouville equation. Starting from the continuity equation for the flow of systems in phase space Γ

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\omega) = 0, \qquad \omega = (q, p) = (q_1, \dots, q_N, p_1, \dots, p_N), \tag{13}$$

where $\rho(p, q, t)$ is the distribution function of the Gibbs ensemble, we obtain the Liouville equation. Since for scalar quantities the covariant derivative coincides with the ordinary one, it follows from (11) and (13) that

$$\frac{\partial\rho}{\partial t} = \{H; \rho\} - \sum_{(j)} Q_j \frac{\partial\rho}{\partial p_j} - \rho \operatorname{div}\omega, \tag{14}$$

where $\{...\}$ are the Poisson brackets.

In calculating $\operatorname{div}\omega$, the differentiation must be covariant, since ω is a vector and the spaces are non-Euclidean.

We shall use the known relation

$$\sum_j \frac{\delta A_{aj}}{\delta q_{aj}} = \frac{1}{\sqrt{-g}} \sum_{(1\le a\le 3)} \frac{\partial}{\partial q_{aj}} \left(\sqrt{-g}\, A_{aj}\right), \tag{15}$$

where $A = (A_{1j}, A_{2j}, A_{3j})$, and

$$g = \det \|g_{jk}\| = -(1 + 2U_e c^{-2})^3 + o(c^{-3}). \tag{16}$$

In momentum space an analogous formula holds, with

$$f = \det \|f_{jk}\| = (1 - p_j^2 m^{-2} c^{-2})^{-3} + o(c^{-3}). \tag{17}$$

After calculations we obtain

$$\operatorname{div}\omega = \frac{3}{mc^2} \sum_{(a,j)} P_{aj} \left(\frac{\partial U_e}{\partial q_a}\right) + o(c^{-3}). \tag{18}$$

Let us examine in more detail two cases: a) the interaction is purely gravitational $(W = U)$; b) the gravitational field is absent $(U = 0)$.

Substituting (18) into (14) and performing some transformations, we arrive at the Liouville equation in the form

$$\frac{\partial \rho}{\partial t} = \{H; \rho\} - \frac{3}{2m^2c^2} D, \tag{19}$$

where

$$D = \begin{cases} \sum_{(i,j)} \frac{\partial}{\partial p_j} \left[p_j^2 \frac{\partial \Phi_{ij}}{\partial q_j} \rho \right], & \text{in case a)}, \\ \sum_{(i,j)} \frac{\partial \Phi_{ij}}{\partial q_j} p_j^2 \frac{\partial \rho}{\partial p_j}, & \text{in case b)}. \end{cases} \tag{20}$$

Chains of Bogoliubov equations. From equations (19), (20), by a straightforward generalization of N. N. Bogoliubov's reasoning, one can obtain equations for the s-particle correlation functions F_s. An essential point in the reasoning is the assumption that the functions ρ tend to zero on an infinitely distant sphere in the space Γ. The equations have the form

$$\frac{\partial F_s}{\partial t} = \{H_s; F_s\} + \frac{1}{v} \iint d\gamma_{s+1} \left\{ \sum_{(1 \leq j \leq s)} \Phi_{j,s+1}; F_{s+1} \right\} - \frac{3}{2m^2c^2} E(F_s, F_{s+1}), \tag{21}$$

where

$$E(F_s, F_{s-1}) = \sum_{(1 \leq j \leq s)} \frac{\partial}{\partial p_j} \left[p_j^2 \left(\frac{\partial W_s}{\partial q_j} F_s + \frac{1}{v} \int \frac{\partial \Phi_{j,s+1}}{\partial q_j} F_{s+1} \, d\gamma_{s+1} \right) \right]. \tag{22}$$

in case a), and

$$E(F_s, F_{s+1}) = \sum_{(1 \leqslant j \leqslant s)} \left[p_j^2 \left(\frac{\partial W_s}{\partial q_j} \frac{\partial F_s}{\partial p_j} + \frac{1}{v} \iint dy_{s+1} \frac{\partial \Phi_{j,s+1}}{\partial q_j} \frac{\partial F_{s+1}}{\partial p_j} \right) \right] \tag{23}$$

in case b).

In formulas (21)–(23) the following notation has been used: $v = \lim(V_q/N)$; $V_q, N \to \infty$ (not to be confused with velocity); $W_s = \sum_{(1 \leqslant i,j \leqslant s)} \Phi_{ij}$; H_s is given by formula (9), with N in it replaced by s.

For example, the equation for F_1 in the case of a purely gravitational interaction is as follows:

$$\frac{\partial F_1}{\partial t} = -\frac{p_1}{m} \frac{\partial F_1}{\partial q_1} + \frac{1}{v} \iint dy_2 \{\Phi_{12}; F_2\} - \frac{3}{2m^2c^2} \frac{1}{v} \frac{\partial}{\partial p_1} \left(p_1^2 \int dy_2 \frac{\partial \Phi_{12}}{\partial q_1} F_2 \right). \tag{24}$$

To obtain solutions of equations (21), it is expedient to represent F_s in the form of the series

$$F_s = F_s^0 + \frac{1}{c^2} F_s^1 + \ldots \tag{25}$$

The equation for F_s^0 is obtained from (21) if one sets there $E(F_s, F_{s+1}) = 0$, while the equation for F_s^1 has the form:

$$\frac{\partial F_s^1}{\partial t} = \{H_s; F_s^1\} + \frac{1}{v} \iint dy_{s+1} \left\{ \sum_{(1 \leqslant j \leqslant s)} \Phi_{j,s+1}; F_{s+1}^1 \right\} - \frac{3}{2m^2c^2} E(F_s^0, F_{s+1}^0). \tag{26}$$

The equation for F_s^0, except for the fact that the integration is performed in non-Euclidean space, differs from the classical equations of N. N. Bogolyubov for nonideal systems. If all F_k^0 are known, then $E(F_s^0, F_{s+1}^0)$ is also known. Thus, the problem has been reduced to solving equation (26) with the known function $E(F_s^0, F_{s+1}^0)$. It makes no sense to seek $F_s^2, F_s^3, \ldots$, since this would lead to exceeding the accuracy with which we are solving the problem.

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REFERENCES

1. N. N. Bogolyubov, *Problems of Dynamical Theory in Statistical Physics*, Moscow, 1946.
2. N. A. Chernikov, DAN, **144**, No. 1, 89 (1962).
3. I. P. Pavlotskii, Problems of statistical mechanics of nonideal systems. II. Bogolyubov equations for correlation functions. Preprint of the Institute of Applied Mathematics, Academy of Sciences of the USSR, No. 18, 1969.
4. V. A. Fock, *The Theory of Space, Time, and Gravitation*, Moscow, 1955.
5. N. V. Efimov, *Higher Geometry*, Moscow, 1947.

Note: Figure translations are in progress. See original paper for figures.