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Abstract

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ON THE RIGIDITY OF SURFACES WITH BOUNDARY IN A RIEMANNIAN SPACE

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Let S be a surface of positive exterior curvature $K \geq k_0 > 0$, with boundary $\mathcal{L}$, in a Riemannian space R_3. Denote by Λ the set of unit vector fields l along $\mathcal{L}$ that are nowhere tangent to the surface S and to the boundary $\mathcal{L}$. We shall study infinitesimal bendings of the surface S subject on the boundary to the condition of generalized sliding:

$$Ul\big|_{\mathcal{L}} = \sigma(s),$$

where U is the bending field of S; $l \in \Lambda$; $\sigma(s)$ is a prescribed function.

Let Λ_0 be the set of unit vector fields l_0 along $\mathcal{L}$ that belong to the surface and are nowhere tangent to $\mathcal{L}$ at any point of the boundary. For each field l_0, $l_0 \in \Lambda_0$, form the set $\Lambda(l_0)$ of vector fields l_α from Λ whose tangential component is collinear with l_0. We shall call $\Lambda(l_0)$ the normal section of the set Λ in the direction of the field l_0. It is evident that each vector field l_α in the section $\Lambda(l_0)$ is uniquely determined by specifying the angle

$$\alpha(s) = \widehat{l_0, l_\alpha},$$

measured in the positive direction.*

Theorem. Let $S \in C^{3,\mu}$, $\mathcal{L} \in C^{1,\mu}$, $l_0, l_\alpha \in C^{1,\mu}$, $\sigma \in C^{1,\mu}$, $0 < \mu < 1$. Then for every normal section $\Lambda(l_0)$, $l_0 \in \Lambda_0$, one can indicate an α_0, $\alpha_0 > 0$, such that for all vector fields l_α from $\Lambda(l_0)$ satisfying the condition

$$\pi - \alpha_0 < \alpha(s) < \pi,$$

the boundary condition $Ul_0 = \sigma(s)$ is quasi-correct and almost rigid, with three degrees of freedom.

Proof. Introducing on the surface an isothermally conjugate parametrization (u, v), we reduce the investigation of infinitesimal bendings with the boundary condition of generalized sliding to the investigation of solvability, in a domain D of the plane $z = u + iv$, of the following boundary-value problem:

$$\partial_z w + B(z)\overline{w} = 0, \qquad z \in D,$$

$$\mathrm{Re}\left\{\left(\partial_t w + \partial_t \ln \sqrt{g\sqrt{K}^3}\, w\right)\sin\alpha - \cos\alpha\,\overline{\lambda(t)}\,w\right\} = \sigma, \qquad t \in \Gamma, \quad (1)$$

where, without loss of generality, the domain D may be regarded as the unit disk; Γ is the boundary of D; B, K, λ, g are known functions of class $C^{1,\mu}$, $0 < \mu < 1$; $\mathrm{Ind}\,\lambda(t) = +1$. To prove the theorem it is enough to establish that one can indicate an α_0, $\alpha_0 > 0$, such that for all functions $\alpha(s)$ of class $C^{1,\mu}$, $0 < \mu < 1$, satisfying the condition

$$\pi - \alpha_0 < \alpha(s) < \pi, \quad (2)$$

the boundary-value problem (1) is solvable for any function σ of class $C^{1,\mu}$, $0 < \mu < 1$, and the solution depends on three parameters.

* In what follows we use the terminology of the book ([1]), Ch. 5, § 10.

Consider the nonhomogeneous problem

$$\partial_{\bar z}\varphi = F(z), \qquad z \in D; \quad (3)$$

$$\mathrm{Re}\{(\partial_t\varphi(t) + \partial_t \ln\sqrt{g\sqrt{K^3}}\,\varphi(t)) - \mathrm{ctg}\,\alpha\,\overline{\lambda(t)}\,\varphi(t)\} = 0, \qquad t \in \Gamma,$$

where $F(z)$ is a given function of class $C^{1,\mu}(D)$, $0 < \mu < 1$. According to results of B. V. Boyarskii ([2]), the number l of solutions of the homogeneous problem (3) is related to the number l^* of solutions of the adjoint problem by the formula

$$l = l^* + 3, \quad (4)$$

and the nonhomogeneous problem (3) is solvable if and only if the orthogonality conditions for the free term of the nonhomogeneous problem (3) with respect to all solutions of the adjoint problem are satisfied. As shown in ([3]), one can specify an a_1, $a_1 > 0$, such that, under the condition $\alpha(s) \in (\pi - a_1, \pi)$, the homogeneous problem (3) has no more than three linearly independent solutions, and therefore from formula (4) we find $l^* = 0$. Consequently, for $\alpha \in (\pi - a_1, \pi)$, problem (3) admits a three-parameter family of solutions for any function $F(z)$ of class $C^{1,\mu}$, $0 < \mu < 1$, which can be represented in the form

$$\varphi(z) = T_\alpha F + \sum_{k=1}^{3} c_k \varphi_\alpha^k, \tag{5}$$

where the operator T_α depends on α; φ_α^k are linearly independent solutions of the homogeneous problem (3); c_k are arbitrary real constants.

Consider the family of problems

$$\partial_z w + B\bar{w} = 0, \qquad z \in D; \tag{6}$$

$$\operatorname{tg}\alpha \ \operatorname{Re}\{\partial_t w + \partial_t \ln\sqrt{g\sqrt{K^3}}\, w\} - \operatorname{Re}\{\overline{\lambda(t)}\, w\} = 0, \qquad t \in \Gamma;$$

regarding α as an arbitrary function of class $C^{1,\mu}$, $0 < \mu < 1$. We shall establish the existence of an $a_0 > 0$ such that problem (6) has three linearly independent solutions for any function $\alpha(s)$ satisfying the condition

$$\pi - a_0 < \alpha(s) < \pi. \tag{7}$$

The constant a_0 is constructed from the coefficients of problem (6) and does not depend on the function $\alpha(s)$. To prove this fact, we reduce problem (6), with the aid of formula (5), to the integral equation

$$w + T_\alpha(B\bar{w}) = \sum_{k=1}^{3} c_k w_\alpha^k, \tag{8}$$

assuming that $\alpha(s) \in (\pi - a_1, \pi)$. We shall seek solutions of problem (3) satisfying the conditions

$$\varphi(0) = 0, \qquad \operatorname{Re}\{\varphi(0)/\chi(0)\} = \operatorname{Im}\{\varphi(0)/\chi(0)\}. \tag{9}$$

Under these conditions the homogeneous problem (3) has only the zero solution if $\pi - a_1 < \alpha < \pi$. The nonhomogeneous problem (3) under conditions (9) is always uniquely solvable. Indeed, from formula (8) it follows that at the point $z = 0$ the following conditions must be satisfied:

$$T_\alpha F(0) + \sum_{k=1}^{3} c_k \varphi_\alpha^k(0) = 0; \tag{10}$$

$$\operatorname{Re}\left\{\frac{1}{\chi(0)}\left[T_\alpha F(0) + \sum_{k=1}^{3} c_k \varphi_\alpha^k(0)\right]\right\} = \operatorname{Im}\left\{\frac{1}{\chi(0)}\left[T_\alpha F(0) + \sum_{k=1}^{3} c_k \varphi_\alpha^k(0)\right]\right\}.$$

These conditions may be regarded as a linear system of three equations with respect to the real constants c_1, c_2, c_3. Since the determi-

of this system is different from zero (otherwise the homogeneous problem (3) would have a nonzero solution), then the constants c_k are determined from (10) always and uniquely. Therefore the solution of the nonhomogeneous problem (3) under conditions (9) can be represented in the form:

$$\varphi = \widetilde{T}_\alpha F, \tag{11}$$

where $\widetilde{T}_\alpha$ is a homogeneous additive operator. We shall show that the operator $\widetilde{T}_\alpha$ admits the representation

$$\widetilde{T}_\alpha = \widetilde{T}_0 + \widetilde{T}_{1\alpha},$$

where $\widetilde{T}_0$ is a completely continuous operator in $C^{1,\mu}$, $0 < \mu < 1$, independent of α, while the norm of the operator $\widetilde{T}_{1\alpha}$, which depends on α, tends to zero as $\alpha \to 0$ in some specially chosen Banach space, for example in $L_2(D)$. Indeed, consider the problem:

$$\partial_z \varphi_0 = F; \qquad \varphi_0(0) = 0; \qquad \mathrm{Re}\{\chi^{-1}(0)\varphi_0(0)\} = \mathrm{Im}\{\chi^{-1}(0)\varphi_0(0)\}; \tag{12}$$

$$\mathrm{Re}\{\lambda(t)\varphi_0\} = 0, \qquad t \in \Gamma.$$

The solution of this problem exists and is unique for any function F of class $C^{1,\mu}$, $0 < \mu < 1$, and is given by the formula

$$\varphi_0 = \widetilde{T}_0 F,$$

where $\widetilde{T}_0 F \in C^{2,\mu}$, $0 < \mu < 1$.

Let us now establish that the norm of the operator $\widetilde{T}_{1\alpha}$ in the space $L_2(D)$ tends to zero as $\alpha \to 0$. For this purpose consider the difference

$$\psi_\alpha = \varphi_\alpha - \varphi_0,$$

where φ_α and φ_0 are the solutions, respectively, of problems (3), (9) and (12). The function ψ_α is analytic in $\bar{D}$ and is a solution of the boundary-value problem

$$\partial_{\bar{z}} \psi_\alpha = 0, \qquad \psi_\alpha(0) = 0, \qquad \mathrm{Re}\{\chi^{-1}(0)\psi_\alpha(0)\} = \mathrm{Im}\{\chi^{-1}(0)\psi_\alpha(0)\};$$

$$\operatorname{Re}\{(\partial_t\psi_\alpha + \partial_t \ln\sqrt{g\sqrt{K^3}}\ \psi_\alpha) - \operatorname{ctg}\alpha\,\overline{\lambda(t)}\psi_\alpha\}$$

$$= -\operatorname{Re}\{\partial_t\varphi_0 + \partial_t \ln\sqrt{g\sqrt{K^3}}\ \varphi_0\}; \qquad t \in \Gamma.$$

It follows from this that the function $\psi_{1\alpha} = \psi_\alpha \chi^{-1}$ is a solution of the boundary-value problem

$$\partial_{\bar z}\psi_{1\alpha} = 0; \qquad \operatorname{Re}\psi_{1\alpha}(0) = \operatorname{Im}\psi_{1\alpha}(0);$$

$$\operatorname{Re}\{(\partial_t\psi_{1\alpha}\cdot\chi(t) + [\partial_t\chi(t) + \partial_t \ln\sqrt{g\sqrt{K^3}}\chi]\psi_{1\alpha}\}$$

$$-\operatorname{ctg}\alpha\ \operatorname{Re}\{\overline{\lambda(t)}\chi\psi_{1\alpha}\} = -\operatorname{Re}\{\partial_t\varphi_0 + \partial_t \ln\sqrt{g\sqrt{K^3}}\varphi_0\}, \qquad t \in \Gamma.$$

But then the functions $u_{1\alpha} = \operatorname{Re}\psi_{1\alpha}$ and $v_{1\alpha} = \operatorname{Im}\psi_{1\alpha}$ satisfy the relation

$$\iint_D (\nabla u_{1\alpha})^2\,dx\,dy + \operatorname{ctg}\alpha \oint A_1^2 u_{1\alpha}^2\,ds = \oint B_1 u_{1\alpha} v_{1\alpha}\,ds +$$

$$+\frac{1}{2}\oint \partial_s \operatorname{tg}\psi\, u_{1\alpha}^2\,ds + \oint u_{1\alpha}\gamma\,ds,$$

where γ is a known function independent of α.

From this formula it follows that (3)

$$\mu_3(\alpha)\|u_{1\alpha}\|^2_{L_2(\Gamma)} \le (\mu_1+\mu_2)\|u_{1\alpha}\|^2_{L_2(\Gamma)} + c\|u_{1\alpha}\|_{L_2(\Gamma)},$$

where c is a constant independent of α. Consequently,

$$\mu_3(\alpha) \le (\mu_1+\mu_2) + c\|u_{1\alpha}\|^{-1}_{L_2(\Gamma)}.$$

Since as $\|a\|_C \to 0$ the constant $\mu_3(a)$ tends to infinity [(3)], it follows from the last inequality that $\|u_{1\alpha}\|_{L_2(\Gamma)} \to 0$ as $\|a\|_C \to 0$. Further, since $\|u_{1\alpha}\|_{L_2(\Gamma)} = \|v_{1\alpha}\|_{L_2(\Gamma)}$, we have $\|\psi_{1\alpha}\|_{L_2(\Gamma)} \to 0$ as $\|a\|_C \to 0$. Taking into account that $\psi_\alpha = \chi\psi_{1\alpha}$ and $\chi \ne 0$ on Γ, hence we obtain that $\|\psi_\alpha\|_{L_2(\Gamma)} \to 0$ as $\|a\|_C \to 0$. Let us estimate the norm of ψ_α in $L_2(D)$. We have

$$\|\psi_\alpha\|^2_{L_2(D)} = \iint_D |\psi_\alpha|^2\,dx\,dy = \int_0^{2\pi}\int_0^1 |\psi_\alpha(re^{i\varphi})|^2 r\,dr\,d\varphi \le \int_0^1 \left(\int_0^{2\pi} |\psi_\alpha(re^{i\varphi})|^2\,d\varphi\right) dr.$$

From the theory of analytic functions it is known that, for any function ψ_α analytic in the disk $|z| < 1$, the L_p-norms

$$\int_0^{2\pi} |\psi_\alpha|^p d\varphi$$

increase monotonically with r. By virtue of the continuity of ψ_α in the closed disk we have

$$\int_0^{2\pi} |\psi_\alpha(re^{i\varphi})|^2\,d\varphi \le \int_0^{2\pi} |\psi_\alpha(e^{i\varphi})|^2\,d\varphi = \|\psi_\alpha\|^2_{L_2(\Gamma)}.$$

Thus, we obtain the estimate

$$\|\psi_\alpha\|_{L_2(D)} \le \|\psi_\alpha\|_{L_2(\Gamma)},$$

whence it follows that $\|\psi_\alpha\|_{L_2(D)} \to 0$ as $\|a\|_C \to 0$.

Let us now consider the equation:

$$w + \widetilde{T}_0(B\overline{w}) + \widetilde{T}_{1\alpha}(Bw) = 0. \tag{13}$$

Putting $P_0 w \equiv \widetilde{T}_0 B\overline{w}$, $P_{1\alpha} w \equiv -\widetilde{T}_{1\alpha} B\overline{w}$, we rewrite it in the form:

$$w + P_0 w = P_{1\alpha} w. \tag{14}$$

By what has been said, the operator P_0 is completely continuous in $C^{1,\mu}$, $0 < \mu < 1$, and $\|P_{1\alpha}\|_{L_1(D)} \to 0$ as $\|a\|_C \to 0$.

It can be shown that the operator P_0 has an inverse, and then equation (14) is equivalent to the equation $w = \Gamma_0 P_{1\alpha} w$, where the norm of the operator Γ_0 does not depend on α. Choosing α_0 sufficiently small for all $\alpha \in (\pi - \alpha_0, \pi)$, we may assume $\|\Gamma_0 P_{1\alpha}\|_{L_2(D)} \le q < 1$. But then equation (14) has only the zero solution, whence the validity of the theorem follows.

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3. V. T. Fomenko, DAN, 181, No. 6 (1969).

Note: Figure translations are in progress. See original paper for figures.