ON THE REDUCTION OF ONE CLASS OF INTEGRAL EQUATIONS OF THE FIRST KIND TO EQUATIONS OF THE SECOND KIND

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Abstract

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ON THE REDUCTION OF ONE CLASS OF INTEGRAL EQUATIONS OF THE FIRST KIND TO EQUATIONS OF THE SECOND KIND

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Consider the following class of two-dimensional equations of the first kind, to which a number of problems of mathematical physics reduce:

$$\int_S w(g) f(g,p)\, dg = \psi(p), \qquad p \in S. \tag{1}$$

Here S is a certain open surface* (Lyapunov) in three-dimensional space, bounded by a contour L; $w(g)$ is the unknown function; $\psi(p)$ is a given free term having regular** continuous first derivatives along $\overline{S}$; $\overline{S} = S + L$; dg is the surface-area element at the point g; f is a kernel of the form:

$$f(g,p) = Q(g,p)/R_{gp} \quad (Q(g,p) \neq 0, \quad g \in \overline{S}), \tag{2}$$

where $Q(g,p)$ is a function continuous on $\overline{S}$ with respect to g and having regular continuous first derivatives with respect to the coordinates of p along $\overline{S}$, and R_{gp} is the distance between the points g and p.

The aim of the present work is to transform (1) into an integral equation of the second kind with a bounded linear operator and a kernel containing an arbitrary (sufficiently smooth) function of two variables. For this purpose we introduce an auxiliary kernel

$$\hat{f}(q,p) = \hat{Q}(q,p)/R_{qp} \quad (\hat{Q}(q,p) \neq 0, \quad q \in \overline{S}), \tag{3}$$

where $\hat{Q}(q,p)$ is defined in some domain V_S containing $\overline{S}$ $(q, p \in V_S)$, and in it has continuous first mixed derivatives with respect to the coordinates of both variables; otherwise it is arbitrary.

Multiply (1) by the derivative of $\hat{f}$ along the normal to S at the point p and integrate over S:

$$\int_S \frac{\partial \hat{f}(q,p)}{\partial n_p} \int_S w(g) f(g,p)\, dg\, dp = \int_S \psi(p) \frac{\partial \hat{f}(q,p)}{\partial n_p}\, dp, \tag{4}$$

where it is assumed that q lies outside $\overline{S}$ ($q \notin \overline{S}$). Since the double integral on the left is finite, by Fubini' s theorem the order of integration in it may be changed, after which (4) takes the form

$$\int_S w(g) \int_S f(g,p) \frac{\partial \hat{f}(q,p)}{\partial n_p}\, dp\, dg = \int_S \psi(p) \frac{\partial \hat{f}(q,p)}{\partial n_p}\, dp, \qquad q \notin \overline{S}. \tag{5}$$

Introduce the notation

$$G(q,g) = \int_S f(g,p) \frac{\partial \hat{f}(q,p)}{\partial n_p}\, dp, \qquad g \in S, \quad q \in V_S. \tag{6}$$

* It may also consist of a finite number of separate surfaces.
** Satisfying Hölder conditions.

Since f and $\hat{f}$ are determined by expressions (2) and (3), they can be represented in the form

$$f(g,p) = Q(g,g)/R_{gp} + u(g,p), \qquad \hat{f}(q,p) = \hat{Q}(q,q)/R_{qp} + \hat{u}(q,p), \tag{7}$$

where u and $\hat{u}$ are expressed elementarily in terms of Q and $\hat{Q}$.

Substituting (7) into (6), we find

$$G(q,g) = Q(g,g)\hat{Q}(q,q) \int\limits_S \frac{1}{R_{gp}} \frac{\cos(\widehat{n_{pR_{pq}}})}{R_{qp}^2}\, dp + G_0(q,g). \tag{8}$$

Here G_0 is a bounded function.

Since

$$\left| \int_S \frac{1}{R_{gp}} \frac{\cos(\widehat{n_{pR_{pq}}})}{R_{qp}^2}\, dp \right| \ll \frac{A}{R_{qg}},$$

(A is a constant), it follows that

$$G(q,g) = \xi_0(q,g)/R_{qg} + G_0(q,g) \equiv \xi(q,g)/R_{qg}, \tag{9}$$

where ξ_0 and ξ are bounded functions.

For fixed g, (6) is an expression of the type of a double-layer potential and, consequently, represents a function continuously differentiable with respect to q up to S, undergoing on S (for $q \neq g$) a jump discontinuity of the first kind, and as $q \to g$ it grows like R_{qg}^{-1}. Taking this into account, one may conclude (see (9)) that ξ, as a function of q, is continuously differentiable up to S and undergoes a jump when passing through S. The limiting values of ξ as q tends to S from the positive and negative sides are, taking into account (8) and (9),

$$(\xi(q,g))^{\pm} = Q(g,g)\hat{Q}(q,q)\left\{R_{qg}\int\limits_S \frac{\cos\,(\widehat{n_p R_{pq}})}{R_{gp}R_{gp}^2}\,dp \pm 2\pi\right\} + R_{qg}(G_0(q,g))^{\pm}.$$

It is also easy to see that for $q = g$

$$(\xi(q,q))^{\pm} = \pm\eta(q)/2\pi; \qquad \eta(q) \equiv 4\pi^2 Q(q,q)\hat{Q}(q,q). \tag{10}$$

Returning to equality (5), we differentiate it in the direction of the normal to S passing through q, after first substituting into (5), in place of the inner integral, expression (9):

$$\frac{\partial}{\partial n_q}\int\limits_S w(g)\frac{\xi(q,g)}{R_{qg}}\,dg = \frac{\partial}{\partial n_q}\int\limits_S \psi(p)\frac{\partial\hat{f}(q,p)}{\partial n_p}\,dp, \qquad q \in \overline{S}. \tag{11}$$

On the left here stands the normal derivative of an expression of the type of a single-layer potential, and on the right—the normal derivative of an expression of the type of a double-layer potential. Therefore, letting q tend to S from different sides, we find, taking into account the behavior of these potentials and the discontinuity of ξ on S*,

$$\int\limits_S w(g)\frac{\partial}{\partial n_q}\left(\frac{\xi(q,g)}{R_{qg}}\right)^{\pm} dg \mp 2\pi w(q)(\xi(q,q))^{\pm} = \frac{\partial}{\partial n_q}\left(\int\limits_S \psi(p)\frac{\partial\hat{f}(q,p)}{\partial n_p}\,dp\right)^{\pm}, \qquad q \in S. \tag{12}$$

Comparing (6) and (9), and introducing the notation

$$\eta(q)T_{\pm}(q,g) = \frac{\partial}{\partial n_q}\left(\frac{\xi(q,g)}{R_{qg}}\right)^{\pm} = \frac{\partial}{\partial n_q}\left(\int\limits_S f(g,p)\frac{\partial\hat{f}(q,p)}{\partial n_p}\,dp\right)^{\pm}, \tag{13}$$

* Here and below, by $\dfrac{\partial}{\partial n_q}(F)^{\pm}$ is meant the limit of $\dfrac{\partial}{\partial n_q}F$ when q tends to S from the positive or negative side, respectively.

$$\eta(q)\Phi_{\pm}(q) = -\frac{\partial}{\partial n_q}\left(\int_S \psi(p)\frac{\partial \hat{f}(q,p)}{\partial n_p}\,dp\right)^{\pm}. \tag{14}$$

Using them and taking (10) into account, we give the equalities (12) the form

$$w(q) = \int_S w(g)\,T_{\pm}(q,g)\,dg + \Phi_{\pm}(q), \qquad q \in S. \tag{15}$$

Thus two integral equations of the second kind have been obtained. It is easy to show that, for w satisfying (1), the right-hand sides of both equations (15) are equal, although their kernels and free terms are different. When

$$\frac{\partial}{\partial n_q}\hat{Q}(q,q) = 0, \quad q \in S,$$

both equations (15) coincide identically.

If equation (1) has a single solution, then the complete equivalence of (1) and (15) follows from the uniqueness of the solution of (15). The latter can be proved by a method analogous to that given in ([1]), if the auxiliary function is taken, for example, in the form $\hat{Q} = \mathrm{const}$, $\hat{Q} = \exp(-ikR_{qp})$, etc.

As shown in ([2]), a kernel of the type (13) corresponds to an unbounded operator. One can, however, pass from equations (15) to equations with a bounded operator, since the essential singularities are contained both in the term with the kernel and in the free term, and they can be separated and mutually canceled. For this it is necessary to use relation (4) ([2]). From it there immediately follows the equality*

$$\frac{\partial}{\partial n_q}\left(\int_S U\,\frac{\partial}{\partial n}\frac{1}{R_{qp}}\,dp\right)^{\pm} = \left(\int_S [\mathbf{n}_q[\mathbf{n}\nabla U]]\,\nabla\frac{1}{R_{qp}}\,dp\right)^{\pm} - \mathbf{n}_q\,\mathrm{rot}_q \oint_L U\,\frac{1}{R_{qp}}\,d\mathbf{L}_p. \tag{16}$$

It, like (4) ([2]), is also valid for multiply connected surfaces S, if by L one understands the total boundary. It follows from this that (16) is also satisfied for functions $U(p)$ having isolated singular points on S, provided only that the contour integral (see (16)), taken around such a point, tends to zero when contracted to it. For example, for $U(p) \equiv f(g,p)$, where $g \in S$ is regarded as a fixed parameter, (16) is valid. It is not difficult, with the aid of (16), to write the expressions (13) and (14) in the form

$$T_{\pm}(q,g) = \hat{T}_{\pm}(q,g) - \frac{1}{4\pi^2 Q(q,q)}\,\mathbf{n}_q\,\mathrm{rot}_q \oint_L \frac{f(g,p)}{R_{qp}}\,d\mathbf{L}_p,$$

$$\Phi_{\pm}(q) = \hat{\Phi}_{\pm}(q) + \frac{1}{4\pi^2 Q(q,q)}\,\mathbf{n}_q\,\mathrm{rot}_q \oint_L \frac{\psi}{R_{qp}}\,d\mathbf{L}_p. \tag{17}$$

Here

$$\hat{T}_{\pm}(q,g) = \frac{1}{\eta(q)}\int_S f(g,p)\frac{\partial^2 \hat{u}(q,p)}{\partial n_q \partial n}\,dp + \frac{1}{\eta(q)}\frac{\partial \hat{Q}(q,q)}{\partial n_q}\left(\int_S f(g,p)\frac{\partial}{\partial n}\frac{1}{R_{qp}}\,dp\right)^{\pm} +$$

$$+\frac{1}{4\pi^2 Q(q,q)}\left(\int_S [\mathbf{n}_q[\mathbf{n}\nabla f(g,p)]]\,\nabla\frac{1}{R_{qp}}\,dp\right)^{\pm}, \tag{18}$$

$$\hat{\Phi}_{\pm}(q) = -\frac{1}{\eta(q)}\int_S \psi\,\frac{\partial^2 \hat{u}(q,p)}{\partial n_q \partial n}\,dp - \frac{1}{\eta(q)}\frac{\partial \hat{Q}(q,q)}{\partial n_q}\left(\int_S \psi\,\frac{\partial}{\partial n}\frac{1}{R_{qp}}\,dp\right)^{\pm} -$$

$$-\frac{1}{4\pi^2 Q(q,q)}\left(\int_S [\mathbf{n}_q[\mathbf{n}\nabla\psi]]\,\nabla\frac{1}{R_{qp}}\,dp\right)^{\pm}. \tag{19}$$

* Here and below we do not put indices on differentiation operators with respect to the coordinates of the point of integration, and also do not write the arguments of functions depending only on the point of integration.

Substituting expression (17) for the kernels and free terms into equations (15), we find

$$w(q) = \int_S w(g)\,\hat{T}_{\pm}(q,g)\,dg + \hat{\Phi}_{\pm}(q), \qquad q \in S. \tag{20}$$

In this case the terms with contour integrals appearing in (17) cancel each other, owing to the fact that w and ψ are related by relation (1). The auxiliary function $\hat{Q}(q,p)$, entering into the kernels and free terms of equations (15) and (20), may be chosen quite arbitrarily. This can be used to simplify the equations themselves and to accelerate the convergence of the processes used to solve them. For example, setting it equal to unity ($\hat{Q} = 1$, and hence $\hat{u} \equiv 0$), we obtain, instead of (18) and (19), the following simplified expressions for the kernel and the free term *:

$$\hat{T}(q,g) = \frac{1}{4\pi^2 Q(q,q)}\left(\int_S [\mathbf{n}_q[\mathbf{n}\nabla f(g,p)]]\,\nabla\frac{1}{R_{qp}}\,dp\right)^{\pm}, \tag{18a}$$

$$\hat{\Phi}(q) = -\frac{1}{4\pi^2 Q(q,q)} \left(\int_S [\mathbf{n}_q[\mathbf{n}\nabla\psi]] \nabla \frac{1}{R_{qp}}\, dp \right)^{\pm}. \tag{19a}$$

Another choice of the function $\hat{Q}$ is also possible.

For $Q = \frac{1}{4\pi}\exp(-ikR_{gp})$, equation (15), and consequently also (20), reduces to the diffraction problem for a scalar wave on the screen S under boundary conditions of Dirichlet type. If, in addition, one sets $\hat{Q} = Q$, then equation (15) coincides with equation (8) of [3], obtained for this problem by another method. However, the kernel (18a) and free term (19a) found here for the equation with bounded operator are considerably simpler than those given in [2].

By a method analogous to that used in [2], it can be shown that the solution of equations (20) should be sought in the class of functions continuous on S and satisfying the Meixner conditions (for the "current" in the Dirichlet problem) on the contour L. The operator $\hat{T}$, corresponding to the kernel $\hat{T}(q,g)$, maps functions of this class into themselves.

The results obtained here extend in an elementary way to one-dimensional integral equations of the first kind in which the integration is over some open curve and the kernel has a logarithmic singularity.

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CITED LITERATURE

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[2] Ya. N. Fel'd, I. V. Sukharevskii, Radio Engineering and Electronics, 12, No. 10 (1967).
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* Both kernels and free terms now coincide, and therefore we omit the indices ± on them.

Note: Figure translations are in progress. See original paper for figures.