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Abstract

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ON CONDITIONS FOR THE SOLVABILITY OF PSEUDODIFFERENTIAL EQUATIONS

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G. Lewy ([1]) constructed an example of a differential equation $P(x, D)u = f(x)$ with infinitely differentiable (even analytic) coefficients, having no solution in the class of generalized functions for most functions $f(x)$ from C^∞. L. Hörmander proved in ([2], [3]) that an analogous property is possessed by pseudodifferential operators (in particular, differential ones) for which one can find a characteristic point $(x, \xi) \in \Omega \times S^{n-1}$ such that

$$p^0(x,\xi) = 0, \qquad \operatorname{Im} \sum_{j=1}^{n} \overline{\frac{\partial p^0(x,\xi)}{\partial \xi_j}} \, \frac{\partial p^0(x,\xi)}{\partial x_j} > 0,$$

where $p^0(x, \xi)$ is the principal part of the symbol of the operator P.

In the present paper we obtain a more general condition, expressed in terms of repeated commutators of the operators P and P^*. In the special case of first-order differential operators, such conditions were obtained by L. Nirenberg and F. Treves in ([4]).

In this paper we also prove the invariance of the principal part of the symbol $p^0(x, \xi)$ of the pseudodifferential operator P with respect to so-called canonical homogeneous transformations (see ([6])). This fact, used by us in the proof of the result mentioned above, is undoubtedly of interest in itself. In particular, it simplifies certain considerations concerning the theory of subelliptic pseudodifferential operators ([3], [7]-[9], [11]).

1. Consider homogeneous canonical transformations of the variables $(x, \xi) \in \Omega \times \{R^n \setminus 0\} \to (x', \xi')\Omega' \times \{R^n \setminus 0\}$ such that

$$x_j = \partial S(x', \xi)/\partial \xi_j, \qquad \xi'_j = \partial S(x', \xi)/\partial x'_j, \qquad (j = 1, \dots, n), \tag{1}$$

where $S(x', \xi)$ is a real function homogeneous of first degree in ξ, and $\det \|\partial^2 S(x, \xi)/\partial x'_R \partial \xi_j\| \neq 0$ for $x \in \Omega,\ |\xi| = 1$.

We shall say that the operator P satisfies condition $(H_{R,N})$ for integers $R, N > 0$, if for every compact set K in Ω one can find a constant C such that for $x \in K$ and $|\xi| = 1$

$$\|\psi\| \leq C \left\{ \left\| \sum_{j=0}^{N} \sum_{|\alpha+\beta|<N} \frac{1}{\alpha!\,\beta!} \frac{\partial^{\alpha+\beta} p_j(x,\xi)}{\partial \xi^\alpha \partial x^\beta}\, y^\beta D^\alpha \psi(y) \lambda^{m-j-(|\alpha|k+|\beta|)/(k+1)} \right\| + \lambda^{m-N/(k+1)} \sum_{|\alpha+\beta|=N} \int \|y^\beta D^\alpha \psi\|\, \lambda^{-|\alpha|(k-1)/(k+1)} \right\}, \quad (2)$$

whatever the function $\psi(y) \in C_0^\infty(R^n)$ and $\lambda \geq 1$. Here $\|\psi\| = \|\psi\|_{L_2(R^n)}$, the function $p_j(x,\xi)$ is positively homogeneous of degree $m-j$ in ξ, $p_0(x,\xi) = p^0(x,\xi)$, and $\sum_{j=0}^{\infty} p_j(x,\xi)$ is the symbol of the operator P.

Theorem 1. *For every pseudodifferential operator P of order m and every generating function $S(x',\xi)$, one can indicate a pseudodifferential operator Q of the same order such that $q^0(x',\xi') =$*

$= p^0(x,\xi)$, where $(x,\xi) \to (x',\xi')$ is the transformation defined by formulas (1), and the condition $(H_{R,N})$ is satisfied for the operator P if and only if it is satisfied for the operator Q.

Let us note that, in particular, Theorem 1 implies the invariance of the set of symbols of subelliptic operators with respect to homogeneous canonical transformations (see ([3,6])).

In special cases this result was used by us earlier in ([7,8]). Theorem 1 can be proved with the aid of the following assertion, due to G. I. Eskin:

Theorem. If $l(x,\xi) \in C^\infty$ is a real-valued function homogeneous of first order in ξ, $\det \|\partial^2 l/\partial x_R \partial \xi_S\| \neq 0$, and

$$|D_\xi^\alpha D_x^\beta a(x,\xi)| \leq C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

for all α and β, then the operator

$$Au(x) = (2\pi)^{-n} \int a(x,\xi)\tilde{u}(\xi) e^{-il(x,\xi)}\, d\xi$$

is a bounded operator from H_s to H_{s-m}.

2. Let us consider the question of the possibility of estimates of the form

$$|u|_s \leq C(K)(|Pu|_t + |u|_{s-1}), \qquad u \in C_0^\infty(K) \quad (3)$$

for some real s and t. Here $|u|_s$ is the norm of the function $u(x)$ in the space $H_s(\Omega)$; K is an arbitrary compact set. Without loss of generality, one may assume that the order of the operator P is equal to 1. Let

$$C_1 = [P^*, P] \equiv P^*P - PP^*, \quad C_{j+1} = [P^*, C_j], \quad j = 1, 2, \dots,$$

be a sequence of first-order operators, and let $c_j^0(x,\xi)$ be the corresponding sequence of principal parts of their symbols. If $p^0(x,\xi)=0$, denote by $k=k(x,\xi)$ the number of the first of the functions $c_j^0(x,\xi)$ that is nonzero at the given point.

Theorem 2. If $p^0(x,\xi)=0$ at some point $(x,\xi)\in\Omega\times S^{n-1}$, the number $k(x,\xi)$ is odd, and $c_k^0(x,\xi)<0$, then estimate (3) cannot hold for any real s and t.

Theorem 3. If $p^0(x,\xi)=0$ at some point $(x,\xi)\in\Omega\times S^{n-1}$, the number $k(x,\xi)$ is even, estimate (3) holds for some s,t, and

$$\operatorname{grad}_{a,\xi}\operatorname{Re}p^0(x,\xi)=0,$$

then on the surface $\operatorname{Re}p^0(x,\xi)=0$ one can indicate such a neighborhood of the point (x,ξ) that the function $\operatorname{Im}p^0(x,\xi)$ does not change sign at the points of this neighborhood.

The last assertion follows from Theorem 2. For the case $s>t-m-1$, an assertion analogous to Theorem 2 was proved in ([7],[10]). The proof of Theorem 2 uses the possibility of canonical transformations, the fact that (3) implies the validity of inequality (2) (see ([4],[6])), and the following assertions.

Lemma 1. If $p\le l-1$, then in an arbitrarily small neighborhood of the origin of coordinates in the plane of the variables (x,y) there is a point (x^0,y^0) such that the polynomial

$$P(x,y)=x^{2l-1}+a_0yx^p+\sum_{j=0}^{2l-1}\sum_{k=2}^{2l-1}a_{kj}y_0^{kx^j}\qquad{}^{(a}\neq 0)$$

is equal to 0 at $(x,y)=(x^0,y^0)$, and

$$\partial P(x^0,y^0)/\partial x>0.$$

Lemma 2. Suppose that the conditions of Theorem 2 are satisfied and $k(x_0,\xi_0)=2l-1$. Assume that at every characteristic point from some neighborhood of the point (x_0,ξ_0) we have $c_1^0(x,\xi)\ge 0$. Then for every $\sigma>0$ one can find a function $w(x)$ such that $\operatorname{Im}w(x)$ is positive definite in a neighborhood of the point $x=x^0$, and

$$p^0(x,\xi+\operatorname{grad}w(x))=O(|x-x_0|^\sigma)$$

as $x\to x_0$.

The concluding part of the proof is carried out according to the same scheme as the proof of Theorem 1.3.2 in ([3]).

3. We give the results of this paragraph only for completeness of exposition. They all follow from Theorem 2 in exactly the same way as Theorems 1.4.6 and 1.4.8 follow from Theorem 1.3.2 in ([3]).

Theorem 4. If $p^0(x,\xi) = 0$ at some point $(x,\xi) \in \Omega \times S^{n-1}$, the number $k(x,\xi)$ is odd, and $c_k^0(x,\xi) < 0$, then for any real ρ and ρ' ($\rho' < \rho$) and any neighborhood ω of the point x, one can find a function $u \in H_{\rho'}(\Omega)$ with support in ω such that $Pu \in C^\infty(\Omega)$, but $u \notin H_\rho(\Omega)$.

Theorem 5. If $p^0(x,\xi) = 0$ at some point $(x,\xi) \in \Omega \times S^{n-1}$, the number $k(x,\xi)$ is odd, and $c_k^0(x,\xi) > 0$, then for every neighborhood ω of the point x one can find a function $f \in C_0^\infty(\omega)$ such that there does not exist a generalized function $u \in D'(\Omega)$ for which $Pu = f$ in ω.

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Note: Figure translations are in progress. See original paper for figures.