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Abstract

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A PRIORI ESTIMATES IN $\mathcal{L}_2$ OF SOLUTIONS OF GENERAL BOUNDARY-VALUE PROBLEMS FOR DEGENERATE ELLIPTIC EQUATIONS OF SECOND ORDER

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The results obtained in ([1]) are extended here to a broad class of degenerate elliptic operators of second order, defined in a domain D with smooth boundary.

Let D be a bounded domain of the Euclidean space E_n, and

$$S = \bigcup_{q=1}^{N} S_q$$

the boundary of D, consisting of closed nonintersecting $(n-1)$-dimensional manifolds S_q (components).

Condition 1. Each component S_q of the boundary S belongs to the class C^m $(m \geq 2)$. This means, in particular, that for each point $x^0 \in S_q$ there is a neighborhood $A_{x^0}(\delta)$ and a transformation $T_{x^0}(x) = (t, y)$ $(y \in E_{n-1})$ such that the intersection $A_{x^0}(\delta) \cap D$ is mapped by the transformation T_{x^0} into the half-ball $K_\delta : t^2 + |y|^2 \leq \delta^2,\ t > 0$.

We choose the transformation $T_{x^0}(x)$ in a special way so that, in the new local coordinates (t, y), for points $x \in A_{x^0}(\delta) \cap D$, t is the coordinate of x along the inward normal to S, and $(y_1, y_2, \dots, y_{n-1}) = y$ are the local coordinates on S of the point of S through which this normal passes.

With such a choice of the transformation $T(x)$, the coordinates t of a point x in two local coordinate systems corresponding to the neighborhoods $A_{x^0}(\delta_0)$ and $A_{x^1}(\delta_1)$ $(x^0, x^1 \in S_q)$ will coincide on the intersection of these neighborhoods. Thus, in some d-neighborhood of S_q a function $t = t(x) \in C^m$ will be defined. Together with it, on the intersection of the d-neighborhood with D, the function

$$a(x) = \sum_{i,j=1}^{n} a_{ij}(x)\, \partial_{x_i} t\, \partial_{x_j} t, \tag{1}$$

will be defined, where $a_{ij}(x)$ are the coefficients of the highest derivatives of the differential operator defined in D,

$$\mathcal{L} = \sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^{n} a_i(x)\partial_{x_i} + a_0(x).$$

Condition 2. The coefficients $a_{ij}(x), a_i(x)$ $(1 \le i, j \le n)$ of the operator are real and continuous in $\overline{D}$. The operator $\mathcal{L}$ is elliptic at every interior point $x \in D$. For definiteness we shall assume that the characteristic polynomial

$$\sigma(x, \lambda) = \sum_{i,j=1}^{n} a_{ij}(x)\lambda_i\lambda_j > 0$$

for $x \in D$ and $\lambda \neq 0$. The boundary manifolds S_q are characteristic for $\mathcal{L}$ ($\sigma(x^0, \nu) = 0$ for $x^0 \in S_q$, ν being the inward normal to S at the point x^0).

Considering the operator $\mathcal{L}$ in $A_{x^0}(\delta) \cap D$, after the transformation $T_{x^0}(x) = (t, y)$ we can write it in the form

$$L_0 = \partial_t(a^0_{nn}(t,y)\partial_t) + \sum_{k=1}^{n-1} a^0_{nk}(t,y)\partial_t\partial_{y_k} + \sum_{k,l=1}^{n-1} a^0_{kl}(t,y)\partial_{y_k}\partial_{y_l} + a^0_n(t,y)\partial_t + \sum_{k=1}^{n-1} a^0_k(t,y)\partial_{y_k} + a^0_0(t,y).$$

Obviously,

$$a^0_{nn} = a(T^{-1}_{x^0}(t,y)), \qquad a^0_{nn}(t,y) > 0 \quad \text{for } t > 0 \quad \text{and} \quad a^0_{nn}(0,y) \equiv 0.$$

Condition 3. For every point $x^0 \in S_q$ the limits exist

$$\lim_{(t,y)\to(+0,0)} \frac{a^0_{nn}(t,y)}{a^0_{nn}(t,0)} = 1; \tag{2}$$

$$\lim_{(t,y)\to(+0,0)} \frac{a^0_{nk}(t,y)}{\sqrt{a^0_{nn}(t,0)}} = \gamma_k \qquad (1 \le k \le n-1); \tag{3}$$

$$\lim_{(t,y)\to(+0,0)} \frac{\left|\operatorname{grad}_y a^0_{nn}(t,y)\right|}{\sqrt{a^0_{nn}(t,0)}} = 0.$$

Introducing the function $\alpha(t) = \sqrt{a_{nn}(t,0)}$ and denoting $\alpha\partial_t = \alpha(t)\partial/\partial t$, we write L_0 in the form $L_0 = L'_0 + L''_0$, where

$$L'_0 = \frac{a^0_{nn}(t,y)}{\alpha^2(t)}(\alpha\partial_t)^2 + \sum_{k=1}^{n-1} \frac{a^0_{nk}(t,y)}{\alpha(t)}\alpha\partial_t\partial_{y_k} + \sum_{k,l=1}^{n-1} a^0_{kl}(t,y)\partial_{y_k}\partial_{y_l},$$

and L_0'' contains derivatives of order < 2.

Definition. We shall say that the operator $\mathcal{L}$ is α-elliptic in $\overline{D}$ if it is elliptic in D and if at every point x^0 of the boundary characteristic manifold S_q $(q = 1, 2, \dots, N)$, under conditions (2), (3), the quadratic form in $\eta, \xi = (\xi_1, \xi_2, \dots, \xi_{n-1})$

$$\zeta(\eta, \xi) = \eta^2 + \sum_{k=1}^{n-1} \gamma_k \eta \xi_k + \sum_{k,l=1}^{n-1} a_{kl}^0(0,0)\xi_k \xi_l \neq 0 \tag{4}$$

does not vanish for any real $(\eta, \xi) \neq 0$. Since we have agreed to assume $\sigma(x, \lambda) > 0$, it follows from (4) that $\zeta(\eta, \xi) > 0$ for $(\eta, \xi) \neq 0$.

Let a boundary differential operator $\mathcal{B}_q$ be given on S_q, which in the neighborhood $A_{x^0}(\delta)$ can be written in the form

$$B_0(-\sqrt{-1}\,\partial_y, \partial_t) = \sum_{s=0}^{r} \sum_{|\tau| \leq \rho_s} b_{\tau,s}(y)(-\sqrt{-1}\,\partial_y)^\tau \partial_t^s,$$

where $\tau = (\tau_1, \tau_2, \dots, \tau_{n-1})$, $|\tau| = \tau_1 + \tau_2 + \dots + \tau_{n-1}$, and $b_{\tau,s}(y)$ are sufficiently smooth complex-valued functions.

Definition. We shall say that the degenerate order of $\mathcal{B}_q$ at the point $x^0 \in S_q$ is equal to m^*, if the degree of the polynomial

$$\sum_{s=0}^{r} \sum_{|\tau| \leq \rho_s} b_{\tau,s}(0) \xi^\tau \eta^{2s}$$

is equal to m^*.

For every point $x^0 \in S_q$ define the functions $b_\mu(t) = a_n^0(t,0) + \mu \partial_t a_{nn}^0(t,0)$, where μ is a real parameter, and construct the homogeneous

polynomial in ξ of degree m^*

$$\vartheta_{x^0}(\xi) = \sum_{s=1}^{r} \frac{b_0^s(0)}{b_1(0) b_2(0) \dots b_s(0)} \Lambda_s(\xi) \left(-\frac{c(\xi)}{b_0(0)}\right)^s + \Lambda_0(\xi),$$

where

$$\Lambda_s(\xi) = \sum_{|\tau| = m^* - 2s} b_{\tau,s}(0) \xi^\tau, \qquad c(\xi) = -\sum_{k,l=1}^{n-1} a_{kl}^0(0,0) \xi_k \xi_l.$$

Definition. We shall say that the boundary operator $\mathcal{B}_q$ of degenerate order m^* satisfies at the point $x^0 \in S_q$ the condition of complementarity with respect to $\mathcal{L}$, if for any $\xi \neq 0$, $\vartheta_{x^0}(\xi) \neq 0$.

Condition 4. We shall assume that for the given integer $m \geq 2$ the components S_q of the boundary S can be divided into three sets (respectively $Q_1, Q_2(m)$, and $Q_3(m)$) according to the following rule:

I. $q \in Q_1$, if $b_0(0) > 0$ for every point $x^0 \in S_q$.

II. $q \in Q_2(m)$, if $b_{\frac{1}{2}(m-1)}(0) < 0$ for every point $x^0 \in S_q$.

III. $q \in Q_3(m)$, if $b_0(0) < 0$ and there exists an integer $\widetilde{m}_q \geq 1$ such that $b_{\frac{1}{2}(\widetilde{m}_q-1)}(0) < 0$, $b_{\frac{1}{2}\widetilde{m}_q}(0) \geq 0$ for all $x^0 \in S_q$, and moreover $m \geq \widetilde{m}_q + 2$ if $\widetilde{m}_q$ is even, and $m \geq \widetilde{m}_q + 3$ if $\widetilde{m}_q$ is odd.

As shown in ([2,3]), one can give a criterion for membership in $Q_1, Q_2(m), Q_3(m)$ that does not depend on the choice of a particular coordinate system at the point x^0. We note that any of the sets $Q_1, Q_2(m), Q_3(m)$ may be empty.

Condition 5. Let, for an integer $m \geq 2$, the coefficients $a_{ij}(x)$ belong to $C^{m-1}(\overline{D})$; $a_i(x)$ $(0 \leq i \leq n)$ belong to $C^{m-2}(\overline{D})$; the coefficients of the boundary operators $\mathcal{B}_q$ (of degenerate order m_q^*) belong to $C^{m-m_q^*}$ (where they are defined).

On the set $C^m(K_d)$ define the norm

$$\|v\|_{m,a}^2 = \sum_{r+2s+|\tau| \leq m} \int_{K_d} |a^i(t,y)\, \partial_t^{i+s} \partial_y^{\tau} v(t,y)|^2 \, dt \, dy,$$

where $a = a(t,y)$ is a weight function.

Construct a finite covering of $\overline{D}$, consisting of a strictly interior subdomain D_{δ_0} of the domain D and a system of neighborhoods $A_{x^p}(\delta_p)$, where $x^p \in S$, $p = 1, 2, \ldots, P$. Let $\varphi_p(x)$ $(p = 0, 1, 2, \ldots, P)$ be functions belonging to $C_0^{\infty}(E_n)$ and forming a partition of unity corresponding to this covering,

$$\sum_{p=0}^{P} \varphi_p(x) = 1 \quad \text{in } \overline{D}, \qquad \operatorname{supp} \varphi_0 \subset D_{\delta_0}, \qquad \operatorname{supp} \varphi_p \subset A_{x^p}(\delta_p).$$

Definition. Let $m \geq 1$ be an integer. We shall say that a function $v(x) \in \mathcal{L}_2(D)$, having generalized derivatives in D up to order m, belongs to $H_a^m(D)$, if there exists such a covering $\overline{D}\{D_{\delta_0}, A_{x^p}(\delta_p), p = 1, 2, \ldots, P\}$ that the norm

$$\|v\|_{m,a} = \left\{ \sum_{|\beta| \leq m} \int_D |\partial_x^{\beta}(\varphi_0 v)|^2 \, dx + \sum_{p=1}^{P} \|(\varphi_p v)_p\|_{m,a_p}^2 \right\},$$

is finite, where $(v)_p = v(T_{x^p}^{-1}(t,y))$; $a_p(t,y) = \sqrt{(a)_p}$, and the function $a(x)$ is defined by formula (1).

In the usual way we introduce the boundary norms $\langle\cdot\rangle_{S_q}^{q,m}$ on the manifolds S_q, putting for a finite function $v(y)$ $(y \in E_{n-1})$

$$\langle v\rangle_{E_{n-1},m} = \int_{E_{n-1}} |\xi|^{2m}|\tilde{v}(\xi)|^2\,d\xi,$$

where $\tilde{v}(\xi)$ is the Fourier transform in E_{n-1} of the function $v(y)$.

Theorem. Let, for a given integer $m \geq 2$, the operator $\mathcal{L}$ and the domain D with boundary

$$S = \bigcup_{q=1}^{N} S_q$$

satisfy conditions 1–5. On each boundary manifold S_q, for $q \in Q_2(m)$, let a boundary operator $\mathcal{B}_q$ be given, satisfying condition 5 and having on S_q a pronounced order equal to

$$m_q^* \leq 2\left[\frac{m-2}{2}\right].$$

Then, for the inequality

$$\|v\|_{m,\alpha} \leq c_m\left\{\|\mathcal{L}v\|_{m-2,\alpha} + \sum_{q\in Q_2(m)} \langle\mathcal{B}_q v\rangle_{S_q,\,m-m_q^*-1} + \|v\|_0\right\}, \tag{5}$$

to hold, where $v(x)$ is any function in $H_\alpha^m(D)$, it is necessary and sufficient that the operator $\mathcal{L}$ be α-elliptic in $\overline{D}$ and that the boundary operators $\mathcal{B}_q$ satisfy the complementing condition with respect to $\mathcal{L}$ on S_q for $q \in Q_2(m)$.

One may consider the following boundary value problem: find a function

$$v(x) \in H_\alpha^{m-2}(D)$$

satisfying the conditions

$$\begin{aligned} \mathcal{L}v &= f \quad \text{in } D;\\ \mathcal{B}_q v &= g_q \quad \text{on } S_q \text{ for } q \in Q_2(m), \end{aligned} \tag{6}$$

where $f \in H_\alpha^{m-2}(D)$ and $g_q(\langle g_q\rangle_{S_q,\,m-m_q^*-1} < \infty)$ are prescribed functions.

If, for the given $m \geq 2$, the hypotheses of the theorem under which inequality (5) holds are fulfilled, and if $Q_3(m)$ is the empty set, then the operator corresponding to the boundary value problem (6) is Noetherian.

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Note: Figure translations are in progress. See original paper for figures.