# ALTERNATIVE PROPERTIES OF PHYSICAL SYSTEMS AND SPECTRAL REPRESENTATIONS IN FIELD THEORY

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## Abstract

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# PHYSICS

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# ALTERNATIVE PROPERTIES OF PHYSICAL SYSTEMS AND SPECTRAL REPRESENTATIONS IN FIELD THEORY

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According to von Neumann ([1]), to alternative properties of physical systems, characterized by their complete presence or absence (causality, locality, symmetries, etc.), there correspond projection operators $E$ with values 1 or 0. The development of this approach makes it possible to formalize a number of axioms of quantum field theory and to construct, in a unified way, spectral representations (s.r.) revealing the analytic properties of the most important functions of the theory.

1. Quantities (energy, etc., to which Hermitian operators correspond) and properties may formally be described by equations for the eigenvalues $\lambda$ of an operator $A$:

$$Af(\xi) = \lambda f(\xi). \tag{1}$$

Here a certain set $\lambda$ corresponds to operators of quantities, while for operators of properties $\lambda = 1$. For them (1) becomes the constraint equation $Ef(\xi) = f(\xi)$ ([2]), whose solution is the class of all functions possessing the property corresponding to $E$.

For systems possessing several distinct properties simultaneously, two cases are important ([1]): 1) for $E$ to be fulfilled, a simultaneous fulfillment of a number of conditions $E_k$ is required; then $E = \prod_k E_k$; 2) for $E$ to be fulfilled, it is necessary that at least one of the $E_k$ be fulfilled; hence $E = \bigcup_k E_k$ (definition of the union of projectors: $\bigcup_1^2 E_k = E_1 + E_2 - E_1 E_2$).

Let us single out classes of projectors expressed through Heaviside $\theta$-functions of coordinates or momenta (these will not include projectors of symmetrizations, etc.). For them the Fourier transform $Ef(\xi) = f(\xi)$ leads (with allowance for the properties of Fourier transforms $\tilde{f}(\eta) = \mathcal{E}(\eta)\tilde{f}(\eta)$, where $\mathcal{E}(\eta)$ is a projector in $\eta$-space) to any of the spectral representations adequate to these properties:

$$\tilde{f}(\eta) = \int d\eta' \, \mathcal{E}(\eta - \eta')\tilde{E}(\eta')\tilde{f}(\eta - \eta');$$

$$f(\xi) = \int d\xi' \, \tilde{\mathcal{E}}(\xi')E(\xi - \xi')f(\xi - \xi'). \tag{2}$$

(2) is similar to the integral equations corresponding to (1) with allowance for boundary conditions, the role of which in (2) is played by restrictions on the Fourier transforms (and unitarity). If $\tilde{\mathcal{E}}(\xi)$ or $\tilde{E}(\eta)$ in (2) are Green' s functions of known equations of motion, then (2) can be rewritten in the form of differential equations with a nonconcretizable right-hand side (cf. ([2])). We note that, since $\theta(0)$ is undefined, in (2), for $f(\xi)$ or $\tilde{f}(\eta)$ singular at zero, quasi-local terms ([3]) may arise.

2. Let us consider the properties of locality and causality. Locality of two fields (currents) means that the commutator $C_{12}(x) \equiv \langle[\varphi_1(x_1), \varphi_2(x_2)]\rangle = 0$ for $x_{12}^2 \equiv (x_1 - x_2)^2 < 0$. Consequently, the class of all local fields

is described by the equation:

$$C_{12}(x) = \theta(x_{12}^2)C_{12}(x). \tag{3}$$

To formalize the locality condition for a larger number of fields, let us construct (3) for the multilocal field $\Phi_m(x_1, \dots, x_m)$ ([4]): the commutator $C'_{mn} = \langle[\Phi_m(x_1, \dots, x_m), \varphi_n(x_n)]\rangle = 0$, if all $x_{kn}^2 < 0$; hence $C'_{mn} = \bigcup_1^m \theta(x_{kn}^2)C'_{mn}$. Then, taking instead of $\Phi_m$ the commutator $C_{12}(x)$ and taking (3) into account, for the commutator of three fields $C_{123}(x) = \langle[[\varphi_1(x_1), \varphi_2(x_2)], \varphi_3(x_2)]\rangle$ we obtain

$$C_{123}(x) = \theta(x_{12}^2)\left(\bigcup_{k=1,2} \theta(x_{k3}^2)\right)C_{123}(x) \equiv$$

$$\equiv \theta(x_{12}^2)[\theta(x_{13}^2) + \theta(x_{23}^2) - \theta(x_{13}^2)\theta(x_{23}^2)]C_{123}(x). \tag{4}$$

From (4) for the $n$-fold commutator it follows by induction that

$$C_{12\dots n}(x) = \prod_{k=2}^{n} \bigcup_{l=1}^{k-1} \theta(x_{kl}^2)C_{12\dots n}(x) \equiv L_{1\dots n}(x_1, \dots, x_n)C_{1\dots n}(x). \tag{5}$$

Equations (4) and (5) are the most general and automatically satisfy not only the locality conditions, but also all symmetry requirements (the Jacobi identity, etc.), generalizing and refining the earlier constructions ([5]).

It is curious that (4) and (5) can be obtained in a less formal way: according to (3), $C_{123}(x) = \theta(x_{12}^2)C_{123}(x)$; substituting here the Jacobi identity $C_{123} = -C_{231} - C_{312}$ and taking (3) into account, for $C_{231}$ and $C_{312}$ we obtain

$$C_{123}(x) = -\theta(x_{12}^2)\{\theta(x_{23}^2)C_{231} + \theta(x_{13}^2)C_{312}\}, \tag{6}$$

which satisfies the locality conditions ($C_{123}(x) = 0$ for $x_{12}^2 < 0$ or $x_{13}^2 < 0$ and $x_{23}^2 < 0$) and the symmetry conditions. If, however, $C_{231}$ and $C_{312}$ are written in the form (6) and substituted again into (6), then such an iteration leads to (4).

In contrast to the projectors $L$ in (5), the retardation operators $R$ were used in many works (for example ([6]))

$${}^1R_{2\ldots n} = \prod_{k=2}^{n} \theta(x_{10} - x_{k0}) = \sum_{\text{perm}} \theta(x_{10} - x_{20}) \ldots \theta(x_{n-1,0} - x_{n,0}) \tag{7}$$

(the last equality (7) is proved with the aid of the identity $\theta(\alpha)\theta(\beta) = \theta(\alpha)\theta(\beta)\theta(\alpha + \beta)$). The proper functions of the product $RL$ are the $r$-functions and radiation operators ([6]); thus, using (5) and (7), it is easy to show that

$${}^1r_{2\ldots n}(x) = \prod_{k=2}^{n} \theta(x_{10} - x_{k0})\theta(x_{1k}^2)\, {}^1r_{2\ldots n}(x). \tag{8}$$

3. The axioms of spectrality and positivity of energy make it possible to write constraint equations for the Wightman functions

$$\widetilde{W}_{1\ldots n}(p) = \prod_{k=1}^{n} \theta(p_{k,0} - p_{k+1,0})\theta(p_{k,k+1}^2 - m_{k,k+1}^2)\widetilde{W}_{1\ldots n}(p) \tag{9}$$

and the Fourier transforms of multiple commutators. Thus, from (9) it follows that $\widetilde{C}_{123}(p) \equiv 0$, if $p_{12}^2 < m_{12}^2$ or $p_{13}^2 < m_{13}^2$ and $p_{23}^2 < m_{23}^2$. Hence, by analogy with (4),

$$\widetilde{C}_{123}(p) = \theta(p_{12}^2 - m_{12}^2)[\theta(p_{13}^2 - m_{13}^2) \cup \theta(p_{23}^2 - m_{23}^2)]\widetilde{C}_{123}(p) \quad \text{etc.} \tag{10}$$

4. The obtained constraint equations make it possible to construct directly s.r. (2). The simplest s.r. for the vacuum mean follow from (6) by the method ([2]):

$$\widetilde{C}_{123}(p_1, p_2) = -\pi^{-6} f \frac{d^4q_1\, d^4q_2}{(p_1 - q_1)^4} \left\{ \frac{\widetilde{C}_{231}(q_1, q_2)}{(p_2 - q_2)^4} - \frac{\widetilde{C}_{312}(q_1, q_2)}{(p_1 + p_2 - q_1 - q_2)^4} \right\}. \tag{11}$$

To obtain an s.r. with one spectral function (4), it is necessary to use the Fourier image of the projector

$$\hat{F}_{p,q}[\theta(x^2)\theta(y^2)\theta((x+y)^2)] = \pi^{-9}\int d^4k\,[\bar{\Delta}(k,0)\bar{\Delta}(p-k,0)\bar{\Delta}(q-k,0)]^2 =$$

$$= \int_0^\infty da\,db\,dc\,\delta(p^2-a)\delta(q^2-b)\delta((p+q)^2-c) \equiv$$

$$\equiv \int_0^\infty da\,db\,dc\,\nabla_1^{(3)}(p,q;a,b,c). \quad (12)$$

a function of the Jost-Wightman type (7).

The s.r. for the Fourier image of the $r$-function follow from (8):

$${}_1\tilde{r}_{2\ldots n}(p) = (2\pi i)^{-n}\prod_1^{n-1}\int_0^\infty d\lambda_k\int d^4q_k\,\nabla^{(+)}(p_k-q_k;\lambda_k)\,{}_1\tilde{r}_{2\ldots n}(p) =$$

$$= (-2^5\pi^7)^{1-n}\prod_0^{n-1}\int d^4q_k\,[\Delta_{\text{ret}}(p_k-q_k,0)]^2\,{}_1\tilde{r}_{2\ldots n}(q). \quad (13)$$

(13) can be rewritten in the form of a multiple Dyson representation (cf. (8)), if the transformations ($n=3$) are performed:

$$\nabla^{(\pm)}(q,\lambda) = \int_0^\infty d\nu\,\left\{\varepsilon(q_0)\bar{\nabla}(\nu,\lambda) \pm \frac{1}{2}\nabla_1(\nu,\lambda)\right\}\delta(q^2-\nu);$$

$$\{\Phi_1;\Phi_2;\Phi_3;\Phi_4\} = -\frac{4}{\pi}\left\{\delta'(\nu_1)\delta'(\nu_2); -\frac{\delta'(\nu_1)}{\pi\nu_2^2}; -\frac{\delta'(\nu_2)}{\pi\nu_1^2}; \frac{1}{(\pi\nu_1\nu_2)^2}\right\}{}_1\tilde{r}_{23}(q). \quad (14)$$

Substituting (14) into (13), we obtain ($\Phi_k = \Phi_k(q,\nu_1,\nu_2)$):

$${}_1\tilde{r}_{23}(p) = \int d^4q_1\,d^4q_2\int_0^\infty d\nu_1\,d\nu_2\,\delta[(p_1-q_1)^2-\nu_1]\,\delta[(p_2-q_2)^2-\nu_2]\times$$

$$\times\{\varepsilon(p_{10}-q_{10})\varepsilon(p_{20}-q_{20})\Phi_1 + \varepsilon(p_{10}-q_{10})\Phi_2 + \varepsilon(p_{20}-q_{20})\Phi_3 + \Phi_4\}. \quad (15)$$

To complete the construction of the s.r. (2), it is necessary to expand the right-hand sides of (13) or (15) in commutators or Wightman functions and substitute (9) or (10) into them. Thus, the s.r. (2), taking into account all axioms of field theory, contain a sum of terms with different spectral functions. Similar s.r. in the $x$-representation can be obtained by Fourier transformation of (9) or (10).

Acting on (13) with the operators $(x_k^2 - s_k^2)$ (where $x_{0k}, x_k$, apparently, are quadratic operators of the duration and localization of the interaction, for example (9), and $s_k^2$ are interval operators (?)), these s.r. can be rewritten in the form of "equations of motion."

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*Note: Figure translations are in progress. See original paper for figures.*