MEASURES OF KERNELS OF COMPREHENSION AXIOMS

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Abstract

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MEASURES OF KERNELS OF COMPREHENSION AXIOMS

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Classical logic contains an undecidable set of nonexistence theorems ([1])*, and when it is strengthened by comprehension axioms, antinomies arise inevitably if this set of logical nonexistence theorems contains the negation of any one of the added comprehension axioms or of any conjunction of the latter. In the present article this result receives a new interpretation from the point of view of the notion, introduced here, of the measure of the kernel of a comprehension axiom.

All formulas and all proofs in this article belong to the system K_0, formulated in ([1]), but are without difficulty reproducible, for example, in the logic of the Zermelo–Fraenkel system** and in other formal systems close to it in their means of expression and proof***.

For reasons of convenience we have modified the alphabet of ([1]) at some points. Capital letters of the Latin alphabet P and Q denote free, and lower-case letters of the Greek alphabet $\psi, \varphi, \chi, \theta$ denote bound predicate variables. Indexed capital letters of the Latin alphabet denote possible term constants (predicates).

As metanotations for definite kinds of variables the same symbols are chosen, but marked above with the sign $\sim$. As general metanotations for arbitrary terms, lower-case letters from the beginning of the Gothic alphabet are chosen for free variables and possible constants, and lower-case letters from the end of this alphabet for bound variables. Capital Gothic letters with a square bracket on the right containing variables denote formulas of the system K_0 that depend exactly on the variables indicated in the square bracket.

Let us also recall that the system K_0 has universal term variables, whose range of values includes the range of values of any predicate variable.

The basis of the new point of view from which the comprehension axioms are considered here is the concept of the measure of a kernel. The kernel of the comprehension axiom****

$$(\mathfrak{E}\tilde{\psi})(\mathfrak{p}_1)\dots(\mathfrak{p}_n)(\tilde{\psi}(\mathfrak{p}_1,\dots,\mathfrak{p}_n) \equiv \mathfrak{A}[\mathfrak{p}_1,\dots,\mathfrak{p}_n]) *****$$

is the formula $\mathfrak{A}[\mathfrak{b}_1, \mathfrak{b}_2, \dots, \mathfrak{b}_n]$.

In what follows, solely for simplicity and brevity in writing formulas, all reasoning is carried out for the case of a kernel of the form $\mathfrak{A}[\mathfrak{b}]$. This in fact does not diminish the generality of the consideration, since extension to the general case of a comprehension axiom with kernel $\mathfrak{A}[\mathfrak{b}_1, \mathfrak{b}_2, \dots, \mathfrak{b}_n]$, introducing a k-place predicate (which may depend on parameters, so that $k \leq n$), requires no essentially different constructions.

* This is also true for a number of logical calculi close in strength.

** It is assumed that the proper set-theoretic axioms and axiom schemes of the Zermelo–Fraenkel system are not included in its logic.

*** Therefore one may, if desired, regard the reasoning as being carried out, for example, in the Zermelo–Fraenkel system.

**** In speaking of comprehension axioms, we by no means assume here (unless this is explicitly stated) that they are actually adjoined to the system K_0; only their properties as formulas are at issue.

***** Strictly speaking, a comprehension axiom scheme is written here.

As a heuristically convenient approach, let us first choose an insufficiently precise, but seemingly quite natural, mode of expression, according to which, in the case where there exists a class defined by some property, it may be regarded as a special kind of measure of this property*.

Let us now consider the scheme of convolution axioms

$$(\mathcal{E}\widetilde{\psi})(\mathfrak{b})(\widetilde{\psi}(\mathfrak{b}) \equiv \mathfrak{A}[\mathfrak{b}]). \qquad (\ )$$

The scheme (α) implies two schemes that are substantially weaker:

$$(\mathcal{E}\widetilde{\psi})(\mathfrak{b})(\widetilde{\psi}(\mathfrak{A}) \rightarrow \mathfrak{A}[\mathfrak{b}]), \qquad (\alpha_1)$$

$$(\mathcal{E}\widetilde{\psi})(\mathfrak{b})(\mathfrak{A}[\mathfrak{b}] \rightarrow \widetilde{\psi}(\mathfrak{b})). \qquad (\alpha_2)$$

Let P_0 (respectively Q_0) be one of those values of the predicate variable ψ whose existence would be asserted in (α_1) (respectively in (α_2)). According to (α_1), the predicate P_0 is satisfied** only by those terms that satisfy the formula $\mathfrak{A}[\mathfrak{b}]$, whereas according to ($\alpha_2$), all terms satisfying the formula $\mathfrak{A}[\mathfrak{b}]$ satisfy Q_0.

Generalizing somewhat the mode of expression outlined above, we could say that P_0 is one of the "included measures," while Q_0 is one of the "covering measures" of the kernel $\mathfrak{A}[\mathfrak{b}]$.

From this point of view the question arises under what conditions the existence, in some definite sense, of a maximal included and a minimal covering measure of the kernel $\mathfrak{A}[\mathfrak{b}]$ could be guaranteed.

First of all, it is evident that the existence of such measures cannot be asserted in the system K_0 itself, since it does not include constant terms, and that it is possible only in certain existential extensions of the system K_0, obtained by adjoining to K_0 certain convolution axioms of a special kind. It is natural here to strive for such extensions to be as weak as possible, remaining, so to speak, almost pure logic.

Moreover, it is clear that a necessary condition for the possibility of constructing, for the kernel $\mathfrak{A}[\mathfrak{b}]$, such convolution axioms of a special kind is the existence, in the system K_0 itself, of formulas that would in this case play the role of the corresponding kernels.

It turns out that for every kernel $\mathfrak{A}[\mathfrak{b}]$ one can, in fact, indicate (up to equivalence) two formulas with the properties needed for the kernels of convolution axioms that introduce the maximal included and the minimal covering measures.

Indeed, consider the two dual formula schemes

$$\mathfrak{M}_i^{\mathfrak{A}}[\mathfrak{b}] \underset{\text{Def}}{=} (\mathcal{E}\widetilde{\psi})\{(\mathfrak{b})(\widetilde{\psi}(\mathfrak{b}) \to \mathfrak{A}[\mathfrak{b}]) \ \& \ \overline{\psi}(\mathfrak{b})\} \ \ast\ast\ast, \qquad (\mathfrak{M}_i)$$

$$\mathfrak{M}_e^{\mathfrak{A}}[\mathfrak{b}] \underset{\text{Def}}{=} (\widetilde{\psi})\{(\mathfrak{b})(\mathfrak{A}[\mathfrak{b}] \to \widetilde{\psi}(\mathfrak{b})) \to \widetilde{\psi}(\mathfrak{b})\}. \qquad (\mathfrak{M}_e)$$

To justify the choice of these formulas, we present a number of metatheorems of K_0.

$$\vdash (\mathfrak{b})(\mathfrak{M}_i^{\mathfrak{A}}[\mathfrak{b}] \to \mathfrak{A}[\mathfrak{b}]) \ \ast\ast\ast\ast; \qquad \text{(I)}$$

$$\vdash (\mathfrak{b})(\mathfrak{A}[\mathfrak{b}] \to \mathfrak{M}_e^{\mathfrak{A}}[\mathfrak{b}]); \qquad \text{(II)}$$

$$\vdash (\overline{\psi})\{(\mathfrak{b})(\overline{\psi}(\mathfrak{b}) \to \mathfrak{A}[\mathfrak{b}]) \to (\mathfrak{b})(\overline{\psi}(\mathfrak{b}) \to \mathfrak{M}_i^{\mathfrak{A}}[\mathfrak{b}])\}; \qquad \text{(III)}$$

$$\vdash (\widetilde{\psi})\{(\mathfrak{b})(\mathfrak{A}[\mathfrak{b}] \to \widetilde{\psi}(\mathfrak{b})) \to (\mathfrak{b})(\mathfrak{M}_e^{\mathfrak{A}}[\mathfrak{b}] \to \widetilde{\psi}(\mathfrak{b}))\}. \qquad \text{(IV)}$$

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* In doing so, however, we do not mean that a measure of this kind is necessarily extensional in its character.

** The expression "the term $\mathfrak{b}$ satisfies the predicate P_0" means that in the formal system (model) under consideration $P_0(\mathfrak{b})$ is a derivable (true) formula.

*** We omit the square bracket with the variable in the superscript for $\mathfrak{M}_i$ and $\mathfrak{M}_e$ when this cannot cause misunderstanding.

**** The symbol $\vdash \mathfrak{F}$ is read as "$\mathfrak{F}$ is a theorem."

From formulas (I) and (II) it is clear that, if $\mathfrak{M}_i^{\mathfrak{A}}[\mathfrak{v}]$ and $\mathfrak{M}_e^{\mathfrak{A}}[\mathfrak{v}]$ are taken as kernels of convolution axioms, then the axiom with the first of these kernels would introduce an embedded measure, while the axiom with the second would introduce a covering measure of $\mathfrak{A}[\mathfrak{v}]$. Under the same assumption, (III) and (IV) would express, respectively, the maximality of such an embedded measure and the minimality of such a covering measure.

Further (using (I) and (II) in the derivation),

$$\vdash \{(\mathfrak{v})(\mathfrak{M}_i^{\mathfrak{A}}[\mathfrak{v}] \equiv \mathfrak{M}_e^{\mathfrak{A}}[\mathfrak{v}]) \to (\mathfrak{v})(\mathfrak{M}_i^{\mathfrak{A}}[\mathfrak{v}] \equiv \mathfrak{A}[\mathfrak{v}])\}, \tag{V}$$

where, obviously, in the consequent the index i at $\mathfrak{M}$ can be replaced by e. It should be noted that whereas the implication

$$(\mathfrak{v})(\mathfrak{M}_i^{\mathfrak{A}}[\mathfrak{v}] \to \mathfrak{M}_e^{\mathfrak{A}}[\mathfrak{v}])$$

is trivial (by virtue of (I) and (II)), its converse is entirely nontrivial and, of course, is not derivable for K_0.

Let us now adjoin to the axiom system K_0 two schemas of convolution axioms:

$$(\mathfrak{v})(M_i^{\mathfrak{A}}(\mathfrak{v}) \equiv \mathfrak{M}_i^{\mathfrak{A}}[\mathfrak{v}]), \tag{M$_i$}$$

$$(\mathfrak{v})(M_e^{\mathfrak{A}}(\mathfrak{v}) \equiv \mathfrak{M}_e^{\mathfrak{A}}[\mathfrak{v}])^*. \tag{M$_e$}$$

It is clear, by virtue of what was said above, that $M_i^{\mathfrak{A}}$ is the maximal embedded measure, and $M_e^{\mathfrak{A}}$ the minimal covering measure of the kernel $\mathfrak{A}[\mathfrak{v}]$.

Let us now introduce two definitions:

D_1. We shall call the **internal measure of the kernel** $\mathfrak{A}[\mathfrak{v}]$ its maximal embedded measure.

D_2. We shall call the **external measure of the kernel** $\mathfrak{A}[\mathfrak{v}]$ its minimal covering measure.

Denote the existential extension of the system K_0 thus obtained by the symbol

$$E\langle K_0; M_i, M_e\rangle.$$

The consistency of $E\langle K_0; M_i, M_e\rangle$ is easily proved by constructing a model, and, in the case where the kernels $\mathfrak{A}[\mathfrak{v}]$ contain occurrences of atomic formulas only of the form $c(\mathfrak{a})$, $\mathfrak{a} = \mathfrak{b}^{**}$, on a model of two objects V, Λ (we denote it by the symbol $\{\Lambda, V\}$), in which the binary relation R_0 and the identity relation are defined so that the propositions $R_0(V,V)$, $R_0(V,\Lambda)$, $\overline{R_0(\Lambda,V)}$, $\overline{R_0(\Lambda,\Lambda)}$, $V = V$, $\Lambda = \Lambda$, $\overline{V = \Lambda}$, $\overline{\Lambda = V}$ are true in $\{\Lambda, V\}^{***}$.

Main metatheorem. Every formula of the form

$$(\mathcal{E}\tilde{\psi})(\mathfrak{v})(\tilde{\psi}(\mathfrak{v}) \equiv \mathfrak{A}[\mathfrak{v}]) \equiv (\mathfrak{v})(M_e^{\mathfrak{A}}(\mathfrak{v}) \to M_i^{\mathfrak{A}}(\mathfrak{v}))^{****}$$

is a theorem of the system $E\langle K_0; M_i, M_e\rangle$.

The principal conclusion from this theorem: adjoining to the system K_0 a convolution axiom with kernel $\mathfrak{A}_0[\mathfrak{v}]$ is equivalent to carrying out two steps successively. The first step consists in passing from the system K_0 to $E\langle K_0; M_i^{\mathfrak{A}_0}, M_e^{\mathfrak{A}_0}\rangle^{*****}$ and, by virtue of what was said above, cannot have as its consequence the appearance of antinomy.

The second step consists in the additional introduction, as an axiom, of the formula

$$(\mathfrak{v})(M_e^{\mathfrak{A}_0}(\mathfrak{v}) \to M_i^{\mathfrak{A}_0}(\mathfrak{v})),$$

which is precisely, so to speak, the "critical" component of the axio-

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* The introduction of constant symbols is inessential, but convenient, since it makes it possible to abbreviate the notation of formulas and proofs.

** In the logic of the system $Z - F$, every atomic formula has the form $\mathfrak{a} \in \mathfrak{b}$ (?) and corresponds to $b(a)$ in K_0.

*** The sufficiency in this case of the model $\{\Lambda, V\}$ was noted by V. N. Grishin.

**** Here $\mathfrak{A}[\mathfrak{v}]$ contains atomic formulas only of the form $c(\mathfrak{a})$ or $c = \mathfrak{b}$.

***** $E\langle K_0; M_i^{\mathfrak{A}_0}, M_e^{\mathfrak{A}_0}\rangle$ is obtained from K_0 by adding the convolution axiom with kernel $\mathfrak{A}_0[\mathfrak{v}]$.

of convolution. If in $E\langle K_0; M_i^{\mathfrak{A}}, M_e^{\mathfrak{A}}\rangle$ the formula

$$(\mathfrak{B})(M_e^{\mathfrak{A}}(\mathfrak{B}) \to M_i^{\mathfrak{A}}(\mathfrak{B})),$$

is derivable, then an antinomy arises*. One may, in other words, say that in the axiom of convolution, from the point of view of the system $E\langle K_0; M_i^{\mathfrak{A}}, M_e^{\mathfrak{A}}\rangle$, there is contained in implicit form the assertion that the outer and inner measures of the kernel $\mathfrak{A}[\mathfrak{b}]$ coincide**.

Definition 1. The kernel $\mathfrak{A}[\mathfrak{b}]$ is called **measurable in the system** $E\langle K_0; M_i, M_e\rangle$ if in it the formula

$$(\mathfrak{v})(M_e^{\mathfrak{A}}(\mathfrak{v}) \to M_i^{\mathfrak{A}}(\mathfrak{v})).$$

is derivable.

Definition 2. If the kernel $\mathfrak{A}[\mathfrak{b}]$ is measurable in the system $E\langle K_0; M_i, M_e\rangle$, then every inner or outer measure of the kernel $\mathfrak{A}[\mathfrak{b}]$ is called its **measure**.

An example of a kernel not measurable in $E\langle K_0; M_i, M_e\rangle$ may be any Russell kernel***; the simplest Russell kernel is $\overline{P}(P)$.

But there also exist sets of n kernels ($n = 2, 3, ...$) that are not jointly measurable in $E\langle K_0; M_i, M_e\rangle$, although joint measurability in $E\langle K_0; M_i, M_e\rangle$ holds for any set of $n-1$ of these kernels. This case corresponds to the so-called group antinomies, whose existence and examples are considered in paper ([3]).

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CITED LITERATURE

[1] D. A. Bochvar, *Matem. sborn.*, **15** (57), 369 (1944).
[2] A. Fraenkel, I. Bar-Hillel, *Foundations of Set Theory*, Moscow, 1966.
[3] D. A. Bochvar, *Matem. sborn.*, **52** (94), 641 (1960).

* Generalization to the case of a set of convolution axioms presents no difficulty.
** In the sense of the derivability of the equivalence $(\mathfrak{b})(M_e^{\mathfrak{A}}(\mathfrak{b}) \equiv M_i^{\mathfrak{A}}(\mathfrak{b}))$.
*** We call a kernel Russellian if the addition to the system K_0 of a single convolution axiom with this kernel leads to an antinomy.

Note: Figure translations are in progress. See original paper for figures.