SOLUTION OF PROBLEMS OF RIEMANN AND HILBERT TYPE FOR A GENERALIZED CAUCHY-RIEMANN SYSTEM WITH A SINGULAR LINE

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Abstract

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SOLUTION OF PROBLEMS OF RIEMANN AND HILBERT TYPE FOR A GENERALIZED CAUCHY-RIEMANN SYSTEM WITH A SINGULAR LINE

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A systematic study of the generalized Cauchy-Riemann system

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\mu}{y} v = 0, \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0,$$

$$2\frac{\partial w}{\partial \bar{z}} - \frac{\mu}{z - \bar{z}}(w - \bar{w}) = 0, \tag{1}$$

where $z = x + iy$, $\partial/\partial\bar{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$, and $w = u + iv$, with coefficients from $\mathcal{L}_p(D)$, $p > 2$, was carried out in the monograph of I. N. Vekua ([1]). The Hilbert problem is also studied there. The Riemann problem for the generalized Cauchy-Riemann system with discontinuous coefficients was investigated by L. G. Mikhailov ([3]). In the monograph of L. G. Mikhailov ([4]), the generalized Cauchy-Riemann system with coefficients of the form $a(z)/z$, $b(z)/z$ was considered. Under additional smallness-type restrictions on $a(z)$ and $b(z)$, all the main propositions of I. N. Vekua' s theory were transferred to this case.

In the works of I. I. Danilyuk ([7,8]), for equation (1) with $\mu = \pm 1$, a generalized Cauchy formula was found and the Hilbert problem was investigated in the case where the contour consists of a finite number of curves having no common points and no self-intersection points.

S. A. Tersenov ([9,10]) studied boundary-value problems of Hilbert type for a degenerating system whose singular part, by means of a certain change of variables and replacement of the unknown functions, is reduced to system (1), the contour being a semicircle lying in the half-plane $y \geq 0$.

In connection with the theory of p-analytic functions, G. N. Polozhii considered a certain system with a line of degeneration on the imaginary axis, formulated several boundary-value problems of Dirichlet and Neumann type, and reduced them to a boundary-value problem of Hilbert type.

In any domain D that does not contain a segment of the singular line $y = 0$, (1) will be a system with analytic coefficients. Therefore it is clear that all solutions of the class $C^1(D)$ will be analytic and, in particular, infinitely differentiable.

The entire plane of the complex variable z is divided into the left and right half-planes S^- and S^+. Denote by $E(S^+)$ the class of solutions of system (1) that are continuous and symmetrically continuable through $(0, \infty)$, i.e., such pairs of functions $u(x, y)$ and $v(x, y)$ that satisfy system (1) in the first and fourth quadrants and $v(x, 0) = 0$. The class $E(S^+)$ is defined analogously.

As in ([14]), it is easy to prove that any solutions of system (1) from the classes $E(S^+)$ and $E(S^-)$ for $\mu > 0$ can be represented in the form

$$w^{\pm}(z) = \frac{2}{B(\mu/2, \mu/2)} \int_0^1 \frac{(1-\sigma)\,\varphi^{\pm}[x + iy(1-2\sigma)]\,d\sigma}{[\sigma(1-\sigma)]^{1-\mu/2}}, \tag{2}$$

where $\varphi^+(z)$ and $\varphi^-(z)$ are analytic functions of the complex variable z in the domains S^+ and S^-, possessing the property $\varphi^{\pm}(\bar{z}) = \overline{\varphi^{\pm}(z)}$.

Now let $\mu < 0$. Then, using the connection between equation (1) for $\mu > 0$ and $\mu < 0$, it is easy to prove that any solution of (1) in the domains S^+ and S^-, for which $w^+(z)/|y|^{-\mu}$ and $w^-(z)/|y|^{-\mu}$ are symmetrically continuable through segments of the real axis $(0, \infty)$ and $(-\infty, 0)$, can be represented in the form

$$w^{\pm}(z) = \frac{2|y|^{-\mu}}{iB(-\mu/2, -\mu/2)} \int_0^1 \frac{(1-\sigma)\psi^{\pm}[x + iy(1-2\sigma)]\,d\sigma}{[\sigma(1-\sigma)]^{1+\mu/2}}, \tag{3}$$

where $\psi^+(z)$ and $\psi^-(z)$ are analytic functions of the complex variable z in the domains S^+ and S^-.

For system (1) we pose the following boundary-value problems:

Problem R. It is required to find a solution of system (1), $w^{\pm}(z)$, in the domains $S^{\pm}$, which for $\mu > 0$ belongs to the classes $E(S^{\pm})$, and for $\mu < 0$, $w^{\pm}(z)/|y|^{-\mu} \in E(S^{\pm})$, under the boundary conditions

$$w^+(iy) = G(iy)w^-(iy) + g(iy), \qquad \mu > 0, \tag{R}$$

$$w^+(iy) = \widetilde{G}(iy)w^-(iy) + \tilde{g}(iy)|y|^{-\mu}, \qquad \mu < 0, \tag{$\widetilde{R}$}$$

where $G(iy)$, $\widetilde{G}(iy)$, $g(iy)$, $\tilde{g}(iy)$ are prescribed functions of points of the contour satisfying a Hölder condition.

Problem Γ. It is required to find a function $w^+(z) = u^+ + iv^+$, $w(z) \in E(S^+)$ when $\mu > 0$, and $w(z)/|y|^{-\mu} \in E(S^+)$ when $\mu < 0$, which is a solution of system (1) in S^+ and is continuously extendable to the imaginary axis, under the boundary conditions

$$a(iy)u^+(iy) - b(iy)v^+(iy) = c(iy), \qquad \mu > 0, \tag{Γ}$$

$$\tilde{a}(iy)u^+(iy) - \tilde{b}(iy)v^+(iy) = \tilde{c}(iy)|y|^{\mu}, \qquad \mu < 0, \tag{$\tilde{Γ}$}$$

where $a(iy)$, $b(iy)$, $c(iy)$, $\tilde{a}(iy)$, $\tilde{b}(iy)$, $\tilde{c}(iy)$ are prescribed real functions of points of the contour satisfying a Hölder condition.

Solution of Problem R. Taking into account the integral representation (2) and the condition of continuity of $w^{\pm}(z)$ up to the contour, we pass in (2) to the limit as $z \to iy$. Making the substitution $1 - 2\sigma = \tau$, we arrive at the equality

$$w^{\pm}(iy) = \frac{2}{2^{\mu} B(\mu/2, \mu/2)} \int_{-1}^{1} \frac{(1+\tau)\varphi^{\pm}(iy\tau)\, d\tau}{(1-\tau^2)^{1-\mu/2}}. \tag{4}$$

It is easy to prove that $w^+(iy)$ and $w^-(iy)$ are boundary values of an analytic function from the right and from the left.

Thus, problem (R) reduces to the following Riemann problem of the theory of analytic functions. It is required to find a pair of analytic functions $\Phi^+(z)$ and $\Phi^-(z)$ from the domains S^+ and S^-, when on the imaginary axis they are connected by the condition

$$\Phi^+(iy) = G(iy)\Phi^-(iy) + g(iy), \tag{5}$$

where $\Phi^+(iy) = w^+(iy)$, $\Phi^-(iy) = w^-(iy)$.

Using the known theorem on the solvability of the conjugation problem for analytic functions ([2,5]), we find $\Phi^+(z)$ and $\Phi^-(z)$. Passing to the points of the contour, we determine $w^{\pm}(iy)$. Inverting the integral representations (4), as in ([13]), we find:

$$\varphi^{\pm}(iy) = \frac{B(\mu/2, \mu/2)}{B(1-\beta, \beta)} \left[\frac{d}{dy} \int_0^y \frac{G_1^{\pm}(\eta)\, \eta\, d\eta}{(y^2-\eta^2)^{\beta}} + \frac{i}{y}\frac{d}{dy} \int_0^y \frac{G_2^{\pm}(\eta)\, \eta\, d\eta}{(y^2-\eta^2)^{\beta}} \right], \tag{6}$$

where

$$G_1^{\pm}(y) = \frac{2^{m-1+2\beta}}{\beta(1+\beta)\dots(\beta+m-1)} \left(\frac{1}{y}\frac{d}{dy}\right)^m \left(y^{2(m+\beta)-1} u^{\pm}(y)\right), \tag{7}$$

$$G_2^{\pm}(y) = \frac{2^{m-1+2\beta}}{\beta(\beta+1)\dots(\beta+m-1)} \left(\frac{1}{y}\frac{d}{dy}\right)^m \left(y^{2(m+\beta)-1} v^{\pm}(y)\right), \tag{8}$$

$m = [\mu/2]$ is the integer part and $\beta = \{\mu/2\}$ is the fractional part of $\mu/2$,

$$u^{\pm}(y) = \operatorname{Re}\Phi^{\pm}(iy), \qquad v^{\pm}(y) = \operatorname{Im}\Phi^{\pm}(iy). \tag{9}$$

Then $\varphi^{\pm}(z)$ is found by the Cauchy formula.

Thus, the following has been proved.

Theorem 1. Let $\chi = \operatorname{Ind} G(iy)$ and $w^{\pm}(\infty) \neq 0$; let the functions $G(iy)$, $g(iy) \in C_{\alpha}^{[\mu/2]+1}$, $-\infty < y < \infty$, possess the properties $G(-iy) = \overline{G(iy)}$, $g(-iy) = \overline{g(iy)}$. Then for $\chi > 0$ the homogeneous and nonhomogeneous problems (R) are unconditionally solvable, and their solution depends linearly on $\chi + 1$ arbitrary real constants. For $\chi < 0$ the homogeneous problem is not solvable. The nonhomogeneous problem for $\chi < 0$ is uniquely solvable, moreover for $\chi = -1$ unconditionally, and for $\chi < -1$ only under the fulfillment of $-\chi - 1$ conditions

$$\int_{-i\infty}^{i\infty} \frac{g(\tau)\,d\tau}{\chi^+(\tau)(\tau+1)^k} = 0, \qquad k = 2, 3, \dots, \chi, \tag{10}$$

where $w^+(z)$ and $w^-(z)$ are given by formula (2), where $\varphi^{\pm}(z)$ are determined from an integral of Cauchy type with density $\varphi^{\pm}(iy) = \varphi^{\pm}(t)$; φ is determined from (6), (7), (8), (9), and the functions $\Phi^{\pm}(z)$ are solutions of the conjugation problem (5) in the theory of analytic functions. In (10), $\chi^+(\tau)$ is the limiting value of the canonical function of problem (5).

Remark 1. Using the integral representation (3) and the boundary condition $(\widetilde{R})$, we reduce the solution of problem $(\widetilde{R})$ to the solution of problem (R).

Remark 2. Consider system (1) in the upper half-plane. In this case one may pose boundary-value problems of type (R) and $(\widetilde{R})$ on that part of the imaginary axis which lies in the upper half-plane. By continuing the boundary conditions and the solutions of the given equation into the lower half-plane, the solution of these problems is reduced to the solution of problems (R) and $(\widetilde{R})$.

Solution of problem Γ. Proceeding in the same way as in the solution of problem (R), we reduce problem (Γ) to the determination of an analytic function in the right half-plane from the boundary condition

$$a(iy)u(0,y) - b(iy)v(0,y) = c(iy) \qquad (0 < y < \infty), \tag{11}$$

where $w^+(iy) = \Phi^+(iy)$, $\Phi^+(iy) = u(0,y) + iv(0,y)$. Solving this problem and returning to problem (Γ), we obtain the following theorem.

Theorem 2. Let $a^2(iy) + b^2(iy) \neq 0$, $\chi = \operatorname{Ind} \dfrac{a(iy) - ib(iy)}{a(iy) + ib(iy)}$, and let the functions $a(iy)$, $b(iy)$, $c(iy)$ belong to the class $C_\alpha^{[\mu/2]+1}$ and satisfy the conditions $a(-y) = a(y)$, $b(-y) = -b(y)$, $c(-y) = c(y)$. Then for $\chi > 0$ the homogeneous and nonhomogeneous boundary-value problems (Γ) are unconditionally solvable, and their solution depends on $\chi/2+1$ real constants. For $\chi < 0$ the homogeneous problem is not solvable. The nonhomogeneous problem is solvable under the fulfillment of $-\chi - 1$ solvability conditions of the form

$$\int_{-i\infty}^{i\infty} \left(\frac{t-1}{t+1}\right)^k \frac{g(t)\,dt}{(t+1)^2\chi^+(t)} = 0, \qquad k = 0, 1, 2, \dots, \chi - 2,$$

where the solution is given by formula (2), and $\varphi^+(z)$ is found from an integral of Cauchy type with density $\varphi^+(iy)$, which is determined by formulas (6), (7), (8), (9), where $\Phi^+(z)$ is a solution of problem (11) in the theory of analytic functions.

Remark 3. Substituting the integral representation (3) into the boundary condition $(\tilde{\Gamma})$, we reduce the solution of problem $(\tilde{\Gamma})$ to the solution of a problem analogous to (Γ).

Remark 4. Consider system (1) in the first quadrant S_1^+. On S_1^+ one may pose boundary value problems analogous to problems (Γ) and $(\tilde{\Gamma})$. Extending the solution of system (1), $w(z)$, and the boundary functions to the fourth quadrant, we reduce the solution of these problems to problems of type (Γ) and $(\tilde{\Gamma})$.

Remark 5. For the Euler-Poisson-Darboux equations

$$\Delta u + \frac{\mu}{y}\frac{\partial u}{\partial y} = 0$$

with $\mu > 0$, we pose the problem

$$a(iy)\left(\frac{\partial u}{\partial x}\right)_{x=0} + b(iy)\left(\frac{\partial u}{\partial y}\right)_{x=0} = c(iy) \qquad (\;)$$

in the class $C_2(S^+)$ ([11]). The following result has been obtained concerning the solvability of this problem.

Theorem 3. Suppose $a^2(iy) + b^2(iy) \neq 0$, the functions $a(iy)$, $b(iy)$, $c(iy)$ belong to the class $C_\alpha^{[\mu/2]+1}$, and satisfy the conditions $a(-iy) = a(iy)$, $b(-iy) = -b(iy)$, $c(-iy) = c(iy)$. Then for $\chi > 0$ the homogeneous and nonhomogeneous problems (Γ) are unconditionally solvable, and their solution depends on $\chi/2+2$ arbitrary real constants. For $\chi < 0$ the homogeneous problem is not solvable. The nonhomogeneous problem is solvable when $-\chi - 1$ solvability conditions are fulfilled; moreover, the solution is given by an explicit formula.

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Note: Figure translations are in progress. See original paper for figures.