ON CERTAIN $\Delta\Sigma$-OPERATIONS WHOSE BASES ARE OBTAINED BY UNITING THE CHAINS OF THE BASE OF THE ORIGINAL OPERATION

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ON CERTAIN $\Delta\Sigma$-OPERATIONS WHOSE BASES ARE OBTAINED BY UNITING THE CHAINS OF THE BASE OF THE ORIGINAL OPERATION

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Let I be the space of indices of cardinality $\tau = \aleph_\nu$; let τ be a strongly inaccessible cardinal number; let $\mathfrak{N}$ be the basic space whose sets are under study; and let $H = (H_\xi)$ be a family of bases possessing some property H. We denote by $\overline{H}$ the set of all bases lacking the property H. If the space of chains Ξ is a topological space, then we shall call the property of bases H a topological property when the closure $[H_\xi]$ has the same property as H_ξ.

If a base N is given, then we construct a new base

$$HN = \left(\bigcup_{\eta \in H_\xi} \eta \right)_{H_\xi \in H},$$

obtained by uniting chains of subsets H_ξ of the base N possessing the property H. If $\Phi_{HN} \prec \Phi_N$, $\Phi_{(HN)^i} \prec \Phi_N$ for $i \in I$, then the property of bases H will be called N-regular. We clarify the conditions of N-regularity of the following properties of bases: H_p (to contain at least p distinct chains of the base N, $2 \le p < \omega$), $H_{\tau'}$ (to contain at least τ' distinct chains of the base N, where $\tau' = \aleph_\nu \le \tau$).

Let N be the base of some $\Delta\Sigma$-operation. The set of all chains of the base N containing the index i is called the truncated base N^i. Let $\mathfrak{M} = \{N\} \cup \bigcup (N^i)$ and $\tau^* \le \aleph_{\nu+1}$. Denote by $\mathfrak{M}_{\tau^*}$ the family of all intersections of sets of the family $\mathfrak{M}$ in number $< \tau^*$.

Lemma 1. *If $K \ne \emptyset$ and the base N is such that for every $M \in \mathfrak{M}_{\tau^*}$ the inclusion $\Phi_M(K) \subset \Phi_N(K)$ holds, then*

$$\Phi_{M_1 \setminus M_2}(K) \subset \Phi_N(K), \qquad \Phi_{M_1 \setminus \bigcup_\alpha M_1^{i_\alpha}}(K) \subset \Phi_N(K),$$

where $M_1, M_2 \in \mathfrak{M}_{\tau^*}$, $\alpha \in I$ *(the last inclusion was observed by Yu. I. Romanovsky* [4]*).*

We shall call the base of a $\Delta\Sigma$-operation N τ^*-regular if: 1) $(\Phi_N, \bigcup_\tau) \prec \Phi_N$, $(\Phi_N, \bigcap_\tau) \prec \Phi_N$; 2) if $M \in \mathfrak{M}_{\tau^*}$, then $\Phi_M \prec \Phi_N$. We shall call the base of a $\Delta\Sigma$-operation N $(\aleph_0)_\nu$-semiregular for $\nu < \omega_\nu$ if: 1) $(\Phi_N, \bigcup_\tau) \prec \Phi_N$, $(\Phi_N, \bigcap_{\mathrm{card}\,\nu}) \prec \Phi_N$; 2) if $M \in \mathfrak{M}_{\aleph_0}$, then $\Phi_M \prec \Phi_N$.

Lemma 2. *Let N and N^c be rigid bases of $\Delta\Sigma$-operations, $\mathfrak{M} = \{N\} \cup \bigcup(N^i)$. In order that the base M of the operation $\Phi_M \equiv (\nu)T_{\{N\}}$ be τ^*-regular for every $\tau^* \leq \aleph_{\nu+1}$, where $\nu = \omega \cdot \lambda \leq \omega_\nu$, it is sufficient that the following conditions be fulfilled: 1) $((\nu)T_N, \bigcup_\tau) \prec (\nu)T_N$, $((\nu)T_N, \bigcap_\tau) \prec (\nu)T_N$; 2) if $L \in \mathfrak{M}_{\tau^*}$, then $\Phi_L \prec \Phi_N$, or, if the base N satisfies condition 2^0) $(\Phi_N, d) \prec \Phi_N$, $(\Phi_{N^c}, d) \prec \Phi_{N^c}$, or $2'$) $(\nu)T_N \prec (\nu)T_N$, $((\nu)T_{N^c}, d) \prec (\nu)T_{N^c}$, then $\Phi_L \prec (\nu)T_N$. In order that the base M be $(\aleph_0)_\nu$-semiregular for $\nu < \omega_\nu$, it is sufficient that the conditions be fulfilled: 1^*) $((\nu)T_N, \bigcup_\tau) \prec (\nu)T_N$, $((\nu)T_N, \bigcap_{\mathrm{card}\,\nu}) \prec (\nu)T_N$; 2^*) if $L \in \mathfrak{M}_{\aleph_0}$, then $\Phi_L \prec \Phi_N$, or, if the base N satisfies condition 2^0) or $2'$), then $\Phi_L \prec (\nu)T_N$. In order that*

in order that the rigid base M^c be τ^*-regular with respect to a class $K \supseteq \emptyset$ for every $\tau^* \leq \aleph_{\nu+1}$, it is sufficient that conditions 1) and 2^* be fulfilled with respect to this class (the conditions of $\aleph_{\nu+1}$-regularity of the bases M and M^c were proved by Yu. I. Romanovskii [4]).

Theorem 1. If the rigid base of the $\Delta\Sigma$-operation N is $\aleph_0$-regular or $(\aleph_0)_\omega$-semiregular with respect to the class of sets $K \supseteq \emptyset$, then the properties of the bases H_p $(2 \leq p < \omega)$, $H_{\aleph_0}$ are N-regular.

Corollary. If $\mathfrak{A}$ is the rigid base of a $(\upsilon)A$-operation, where $\upsilon = \omega \cdot \lambda \leq \omega_\nu$, then the properties of the bases H_p $(2 \leq p < \omega)$, $H_{\aleph_0}$ are $\mathfrak{A}$-, $\mathfrak{A}^c$-regular.

Theorem 2. If N and N^c are rigid bases of $\Delta\Sigma$-operations, the rigid base M of the operation $\Phi_M \equiv (\upsilon)T_{\{N\}}$ satisfies conditions 1) and 2) of Lemma 2, then the property of the bases $H_{\tau'}$ is M-regular with respect to any class of sets $K \supseteq \emptyset$, where $\tau' = \aleph_\nu \leq \tau$.

Indeed, consider τ' different chains of the base M, where $\tau' < \tau$. Since M is a rigid base, one can indicate τ' different indices, pairwise not subordinate to one another, each of which enters one and only one of the chains under consideration. Such indices can be arranged in sequences of type ω_ν. Let $i_0, \dots, i_\alpha, \dots \mid \omega_\nu$ be one of such sequences. Then

$$\Phi_{H_{\tau'} M}\{E_i\} = \bigcup_{i_0, \dots, i_\alpha, \dots} \bigcap_{p < \omega_\nu} \Phi_{M^{i_p}} {}^{i_p} \bigcup_{\alpha \neq p,\ \alpha < \omega_\nu} M^{i_\alpha}\{E_i\},$$

where the union is taken over all possible sequences of pairwise non-subordinate indices of type ω_ν, belonging to the set ζ of all indices entering the chains of the base M, and (E_i) is a family of sets of the class $K \supseteq \emptyset$. Since the base M, by Lemma 2, is τ^*-regular for every $\tau^* \leq \aleph_{\nu+1}$, it follows, by Lemma 1, that

$$\Phi_M^{i_p} \bigcup_{\alpha \neq p,\ \alpha < \omega_\nu} M^{i_\alpha}\{E_i\} \subseteq \Phi_M(K).$$

Since τ is a strongly inaccessible cardinal number, the number of different sequences of pairwise non-subordinate indices $i_0, \dots, i_\alpha, \dots \mid \omega_\nu$, belonging to ζ, is not greater than $\tau^{\tau'} = \tau$. By condition 1) of Lemma 2, $\Phi_{H_{\tau'}M}(K) \subseteq \Phi_M(K)$. The operation

$$\Phi_{H_\tau M}\{E_i\} = \bigcap_{\tau' < \tau} \Phi_{H_{\tau'}M}\{E_i\}.$$

By condition 1) of Lemma 2, $\Phi_{H_\tau M}(K) \subseteq \Phi_M(K)$. The operations

$$\Phi_{(H_{\tau'}M)^i}\{E_j\} = \Phi_{H_{\tau'}M}\{E_j\} \cap \Phi_{M^i}\{E_j\},$$

$$\Phi_{(H_\tau M)^i}\{E_j\} = \Phi_{H_\tau M}\{E_j\} \cap \Phi_{M^i}\{E_j\},$$

whence it follows that

$$\Phi_{H_{\tau'}M}(K) \subseteq \Phi_M(K), \qquad \Phi_{H_\tau M}(K) \subseteq \Phi_M(K).$$

Corollary. The property of the bases $H_{\tau'}$, for $\tau' = \aleph_\nu \leq \tau$, is $\mathfrak{A}$-regular.

On the basis of the first and second theorems on multiple separation for the class of sets $\Phi_N(K)$ with respect to the operation Φ_{HN}, in the case when the property of the bases H is N-regular, we obtain the first and second theorems on coverings of sets.

First theorem on coverings of sets. If N is a rigid τ^*-regular or $(\aleph_0)_\upsilon$-semiregular base, where $\upsilon = \omega \cdot \lambda < \omega_\nu$, the operation Φ_N is normal, the class of sets $\Phi_N(K)$ has a system of fully regular indices, H is an N-regular property of bases, then for every family of sets (E_i) of the class $\Phi_N(K)$ such that $\Phi_N\{E_i\} \subseteq \Phi_{HN}\{E_i\}$,

there will be found a family of sets (U_i) of the class $B\Phi_N(K)$ such that $U_i \supset E_i$ and $\Phi_N\{U_i\} \subset \Phi_{HN}\{U_i\}$.

The second theorem on covering sets. If N is a rigid τ^*-regular base or an $(\aleph_0)_\nu$-semiregular base, where $\nu = \omega \cdot \lambda < \omega_\nu$, the operation Φ_N is normal, the class of sets $\Phi_N(K)$ possesses a system of regular transfinite indices, and H is an N-regular property of bases, then for every family of sets (E_i) of the class $\Phi_N(K)$ there will be found a family of sets (U_i) of the class $\Phi_{N^c}(K^c)$ such that

$$U_i \supset E_i \setminus \Phi_{(HN)^i}\{E_j\}, \qquad \Phi_{HN}\{U_i\} = \phi.$$

These theorems hold for the properties H_p and $\bar{H}_\tau$, for $\tau' = \aleph_{\nu'} \le \tau$, in the following cases:

1) for the operations $\Phi_{N_{\alpha+1}} \equiv T_N^{\alpha+1}$ with depth of chains ω relative to the class of sets $T_N^{\alpha+1}(K)$, where $K \supset \mathfrak{M}$, N and N^c are rigid bases, the base N satisfies the conditions: 1^0) $T_N \succ \bigcup_{\aleph_0}$, $T_N \succ \bigcap_\tau$, $T_N \succ \Phi_{N^c}$; 2^0) $(\Phi_N, \tilde{\mathcal{A}}) \prec \Phi_N$, $(\Phi_{N^c}, d) \prec \Phi_{N^c}$; 3^0) if $L \in \mathfrak{M}_{\tau^*}$, where $\mathfrak{M} = \{N\} \cup (N^i)$, then $\Phi_L \prec \Phi_N$ or $\Phi_N \prec T_N$ (if N is a rigid base of the operation $\bigcup_{\aleph_0}$, then we obtain the results of A. A. Lyapunov ([1]) and of the author ([3]));
2) for the (ω_ν) H-operations $\Phi_{\mathfrak{A}}$.

A point $x \in \mathfrak{A}$ is called a point of N—p-valuedness (less than N—τ'-valuedness) of a family of sets (E_i), if there exist p and only p distinct chains of the base N (less than τ' distinct chains of the base N) whose kernels contain the point x. For $p = 1$ we obtain points of N-single-valuedness of the given family of sets; for $\tau' = \aleph_0$, points of N-finite-valuedness. We shall clarify the nature of these points for individual operations.

Theorem 3. If N is a rigid base of a $\Delta\Sigma$-operation satisfying conditions $1^0, 2^0, 3^0$; (E_i) is a family of sets of the class $BT_N^{\alpha+1}(K)$; $N_{\alpha+1}$ is a rigid base of an operation of type $T_N^{\alpha+1}$, then the set of points of $N_{\alpha+1}$-single-valuedness of the family of sets (E_i) belongs to the class of sets $CT_N^{\alpha+1}(K)$.

For $\tau = \aleph_0$ the theorem was proved by A. A. Lyapunov ([1]).

Theorem 4. If N is a rigid base of a $\Delta\Sigma$-operation satisfying conditions $1^0, 2^0, 3^0$; $K \supset \phi$; (E_i) is a family of sets of the class $BT_N^{\alpha+1}(K)$; $N_{\alpha+1}$ is a rigid base of an operation of type $T_N^{\alpha+1}$, then the set of points of $N_{\alpha+1}$—p-valuedness of the family of sets (E_i) belongs to the class $CT_N^{\alpha+1}(K)$.

For $\tau = \aleph_0$ we have the theorem of A. A. Lyapunov ([1]).

Theorem 5. If N and N^c are rigid bases, the base N has the properties $1^0, 2^0, 3^0$; $N_{\alpha+2}$ is a rigid base of an operation of type $T_N^{\alpha+2}$; $(E_i)_i$ is a family of sets of the class $BT_N^{\alpha+2}(K)\,[BT_N^1(K)]$, where $K \supset \phi, \mathfrak{A}$, then the set of points of $N_{\alpha+2}$—τ'-valuedness (N_1—τ'-valuedness), for any $\tau' = \aleph_{\nu'} < \tau$ and $\alpha < \omega_{\nu+1}$, belongs to the class $CT_N^{\alpha+2}(K)\,[CT_N^1(K)]$.

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Note: Figure translations are in progress. See original paper for figures.