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ON ONE NEW TYPE OF BIFURCATION OF MULTIDIMENSIONAL DYNAMICAL SYSTEMS

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§ 1. One of the important problems of the qualitative theory of differential equations is the study of the nature of the change in the qualitative structure of the decomposition of phase space into trajectories in passing from one rough system to another. For dynamical systems of the second order in the plane, the main types of bifurcation were completely considered in the works of A. A. Andronov and E. A. Leontovich. The bifurcations in multidimensional space that have been considered up to the present time are connected primarily with the appearance (disappearance) of only one limiting element of the type of an equilibrium state, a periodic motion, or an integral manifold of torus type. These bifurcations are, in a certain sense, related by the following fact: transitions through bifurcation surfaces (with the exception of the birth of a torus with an irrational Poincaré rotation number) do not lead out of the class of systems satisfying Smale' s "five" conditions.

In the present paper a fundamentally new principal type of bifurcation is indicated, connected with the appearance of a countable set of periodic motions of saddle type and a locally disconnected limit set.

§ 2. Consider a system of differential equations of order $(m+n+1)$

$$dx/dt = P(x,y,z,\mu), \qquad dz/dy = Q(x,y,z,\mu),$$
$$dz/dt = R(x,y,z,\mu), \tag{1}$$

where $P(P_1,\dots,P_m)$, $Q(Q_1,\dots,Q_n)$, and R are sufficiently smooth functions of the variables $x=(x_1,\dots,x_m)$, $y=(y_1,\dots,y_n)$, z, and of the parameter μ in some domain $G\times(-\mu_0,\mu_0)$.

Suppose that for $\mu=0$ system (1) has an equilibrium state $O(0,\dots,0)\in G$ of saddle-saddle type, i.e., m $(m\geq 1)$ roots of the characteristic equation have negative real parts, n $(n\geq 1)$ roots have positive real parts, one root is equal to zero, and the first nonzero Lyapunov quantity l_k has even index k. The structure of a neighborhood of such a complex state is known ([1]) and is characterized primarily by the presence of a unique smooth integral half-surface π^+ of dimension $m+1$, consisting of 0^+-curves, a unique smooth integral half-surface π^- of dimension $n+1$, consisting of 0^--curves, and a surface π of dimension $m+n$, in which the boundaries of the half-surfaces π^+ and π^- lie. All other trajectories leave the neighborhood of O both as $t \to +\infty$ and as $t \to -\infty$.

With respect to system (1) for $\mu = 0$ we shall assume that the following properties are satisfied:

1) there exist p trajectories $\Gamma_1, \dots, \Gamma_p$, doubly asymptotic to O;

2) $\Gamma_i \subset \pi,\ i = 1, 2, \dots, p$;

3) π^+ and π^- intersect along Γ_i roughly, i.e.

$$\dim\left(W^+_{M_i} \cap W^-_{M_i}\right) = 1, \qquad i = 1, \dots, p, \tag{2}$$

where by $W^+_{M_i}$ and $W^-_{M_i}$ are denoted the tangent subspaces to π^+ and π^- at the point $M_i \overset{i}{\in} \Gamma_i$.

Let $U(\Gamma_1, \dots, \Gamma_p, \varepsilon)$ be some sufficiently small ε-neighborhood of the set $\Gamma_1, \dots, \Gamma_p$. From the indicated properties of the system it will follow that, apart from the trajectories $\Gamma_1, \dots, \Gamma_p$ and O, $U(\Gamma_1, \dots, \Gamma_p, \varepsilon)$, for any sufficiently small ε, will contain no other trajectories lying wholly in it. Thus, for $\mu = 0$ the system (1), in view of (2) and $l_2 \neq 0$, is the simplest nonrough one (at least in $U(\Gamma_1, \dots, \Gamma_p, \varepsilon)$).

§ 3. **Theorem.** *There exists an ε_0 such that in any neighborhood $U(\Gamma_1, \dots, \Gamma_p, \varepsilon)$, where $0 < \varepsilon \leq \varepsilon_0$, for all sufficiently small μ ($|\mu| < \mu^*(\varepsilon)$) for which the saddle-saddle O disappears, the set of all trajectories lying wholly in $U(\Gamma_1, \dots, \Gamma_p, \varepsilon)$ is in one-to-one correspondence with the set of all sequences, infinite in both directions, composed of p symbols.*

Remark 1. From this theorem it follows that in the space of smooth dynamical systems, systems with a finite number of periodic motions (in particular, without periodic motions) can be separated from systems with a countable set of periodic motions by only one bifurcation locally connected "film" of codimension 1.

Remark 2. For $p = 1$ we obtain the result of [2].

By means of a certain sufficiently smooth change of variables in a small neighborhood of the origin, system (1) can be brought to the form

$$\begin{aligned} d\xi/dt &= A\xi + P'(\xi, \eta, z)\mu + \mu\alpha(z, \mu)R_0(z, \mu), \\ d\eta/dt &= B\eta + Q'(\xi, \eta, z)\xi + \mu\beta(z, \mu)R_0(z, \mu), \\ dz/dt &= R_0(z, \mu) + R_1'(\xi, \eta, z, \mu)\xi + R_2'(\xi, \eta, z, \mu)\eta, \end{aligned} \tag{3}$$

where A is an m-dimensional square matrix whose characteristic roots lie in the left half-plane; B is an n-dimensional square matrix whose characteristic roots lie to the right of the imaginary axis; P', Q', R_1', and R_2' are small for sufficiently small ξ, η, z, and μ; $R_0(z_0, 0) = l_k z^k + \cdots$, where k is even, $l_k > 0$. The condition for the disappearance of the saddle-saddle is equivalent to the fulfillment of the inequality $R_0(z, \mu) > 0$. We shall assume that $(0, \bar{\mu}_0)$ is that interval of values of μ for which $R_0(z, \mu) > 0$, and denote the values μ themselves from $(0, \bar{\mu}_0)$ by $\bar{\mu}$.

As follows from [2], between a certain set $\sigma_0(\mu) \subset S_0$, where $S_0 : [z = -d,\ \|\eta\| < R_0(-d, \bar{\mu}),\ \|\xi\| < R_0(-d, \bar{\mu})]$, and the set $\sigma_1(\mu) \subset S_1$, where $S_1 : [z = d,\ \|\eta\| < R_0(d, \bar{\mu}),\ \|\xi\| < R_0(d, \bar{\mu})]$, one can establish a one-to-one correspondence $T_0(\bar{\mu}) : M_0(\xi_0, \eta_0) \in \sigma_0 \to M_1(\xi_1, \eta_1) \in \sigma_1$. In this case the relation between the coordinates of the points M_0 and M_1 will be

$$\xi_1 = \xi(\xi_0, \eta_1, \bar{\mu}), \qquad \eta_0 = \eta(\xi_0, \eta_1, \bar{\mu}),$$

where ξ and η are defined for $\|\xi_0\| \leq a < R_0(-d, 0)$, $\|\eta_1\| \leq b < R_0(d, 0)$, and sufficiently small μ, and tend to zero as $\bar{\mu} \to 0$ together with the first derivatives with respect to ξ_0 and η_1.

Let us note that $\eta_0 = 0$ is the equation of the intersection of π^+ with S_0, while $\xi_1 = 0$ is that of π^- with S_1.

It can be shown that, for sufficiently small d, the trajectories $\Gamma_1, \dots, \Gamma_p$ of system (1) for $\mu = 0$ will intersect the surfaces S_0 and S_1. Let $M_0^i(\xi_0^i, 0)$ and $M_1^i(0, \eta_1^i)$ be the points of intersection of Γ_i with S_0 and S_1, and let $T_i(\bar{\mu})$ be the mapping of a neighborhood U_i^1 of the point M_1^i onto a neighborhood U_i^0 of the point M_0^i, induced by trajectories close to Γ_i for sufficiently small μ. Moreover, since π^+ and π^- intersect transversally along Γ_i, the equations $T_i(0)(\pi^- \cap U_i^1)$ can be written in the form

$$\xi_0 = F_0^i(\eta_0), \tag{4}$$

where F_0^i is a smooth function, defined for $\|\eta_0\| \leq a_i$. Since the mapping

the mapping $T_0(\bar{\mu})$ stretches in the direction η_1 and contracts in the direction ξ_1, it is easily shown that $T_0(\bar{\mu})(U_i^0 \cap \sigma_0(\bar{\mu})) \cap (U_j^1 \cap \sigma_1(\bar{\mu})) \neq \emptyset$ for sufficiently small $\bar{\mu}$, for all $1 \leq i, j \leq p$. Denote by $T_{ij}(\bar{\mu})$ the corresponding restriction of $T_0(\bar{\mu})$. Now it is easy to see that, for any $1 \leq i, j \leq p$, each set $U_i^0 \cap \sigma_0(\bar{\mu})$, under the mapping $T_j(\bar{\mu}) T_{ij}(\bar{\mu})$, has a nonempty intersection with $(U_j^0 \cap \sigma_0(\bar{\mu}))$. The indicated situation is very close to the construction described by Smale ([3]), which leads to symbolic dynamics with a finite number of symbols. To establish this we shall prove the existence of two families of transverse foliations. We note that the construction of such families is, in spirit, close to the method of the paper ([4]).

Introduce p metric spaces $H^0_{1a}, \dots, H^0_{pa}$, where H^0_{ia} is the space of n-dimensional surfaces P_i, with the C^0 metric, whose equations are written in the form

$$\xi_0 = \varphi_0(\eta_0), \tag{5}$$

where $\varphi(\eta_0)$ is defined for $\|\eta_0\| \le a < a_i$, $\|\varphi_0(\eta_0) - \xi^0_i\| < \Delta$, and the Lipschitz constants do not exceed the quantity K, where

$$K > \left\{ \max_{\|\eta_0\| \le a} \|\partial F^1_i / \partial \eta_0\| + 1,\ 1 \le i \le p \right\}.$$

Lemma 1. There exists an $\bar{a} \le a$ such that, for all sufficiently small $\bar{\mu}$, on the spaces $H^0_{i\bar{a}}$, for any $1 \le i, j \le p$, operators $\mathcal{T}_{ji}$ are defined satisfying the following conditions:

1) $\mathcal{T}_{ji} H^0_{i\bar{a}} \subset H^0_{j\bar{a}}$;
2) $\rho(\mathcal{T}_{ji} P'_i, \mathcal{T}_{ji} P''_i) < q\rho(P'_i, P''_i)$, where $q < 1$.

Let

$$\nu = (\dots j_{-l}, \dots j_0, \dots j_k, \dots) \tag{6}$$

be an arbitrary infinite sequence in both directions, composed of the symbols $1, 2, \dots, p$. To the sequence ν we assign the sequence of spaces and operators

$$\dots \to H^0_{j_{\rho-1} a} \xrightarrow{\mathcal{T}_{j_{\rho-1} j_\rho}} H^0_{j_\rho a} \xrightarrow{\mathcal{T}_{j_\rho j_{\rho+1}}} H^0_{j_{\rho+1} a} \to \dots \tag{7}$$

Using the lemma on the existence of a stable fixed point in the direct product of spaces (see ([5]), as well as ([4]), where its analogous use is indicated), we derive the following lemma:

Lemma 2. To the sequence ν there corresponds a unique stable sequence of surfaces

$$\dots P^\nu_{j_{-l}}(\bar{\mu}), \dots,\ P^\nu_{j_0}(\bar{\mu}), \dots,\ P^\nu_{j_m}(\bar{\mu}) \dots \tag{8}$$

such that

$$T^{-1}_{j_{\rho+1} j_\rho}(\bar{\mu})\, T^{-1}_{j_{\rho+1}}(\bar{\mu})\, P^\nu_{j_{\rho+1}}(\bar{\mu}) \subset P^\nu_{j_\rho}(\bar{\mu}) \tag{9}$$

for any indices j_ρ, $\rho = 0, \pm 1, \pm 2, \dots$, and where $P^\nu_{j_\rho}(\bar{\mu})$ tends to $T_{j_\rho}(0)(U^1_{j_\rho} \cap \pi^-)$ as $\bar{\mu} \to 0$.

In a similar way one can establish that to any sequence ν there corresponds a unique sequence, stable in the negative direction, of m-dimensional surfaces $\{Q_{i_\rho}^{\nu}\}$, whose equations are written in the form

$$\eta_0 = \psi_{i_\rho}^{\nu}(\xi_0, \bar{\mu}), \tag{10}$$

where the function $\psi_{i_\rho}^{\nu}$ is defined for $\|\xi_0 - \xi_0^{i_\rho}\| \leq \bar{b}$ ($\bar{b}$ is some constant) and, as $\bar{\mu} \to 0$, tends to zero together with the Lipschitz constant.

Since the surfaces $P_{i_\rho}^{\nu}(\mu)$ and $Q_{i_\rho}^{\nu}(\bar{\mu})$ will intersect at the single point $M_{i_\rho}^{\nu}(\bar{\mu})$, it follows that the sequences $\{M_{i_\rho}^{\nu}(\bar{\mu})\}$ will correspond to a phase trajectory lying entirely in $U(\Gamma_1 \ldots \Gamma_p, \varepsilon)$. It is easy to see that the trajectories found in this way exhaust the set of all trajectories lying entirely in $U(\Gamma_1 \ldots \Gamma_p, \varepsilon)$ for sufficiently small ε.

Denote by Σ the set of intersection points of trajectories lying entirely in $U(\Gamma_1 \ldots \Gamma_p, \varepsilon)$ with S_0, endowed with the topology naturally induced from the Euclidean topology. The mappings $T_{jT_{ji}}$ generate on Σ a mapping, which we denote by T. It is easy to see that the discrete system (T, Σ) is topologically equivalent to the system (Ω_p, ω), where Ω_p is the Tikhonov product of a countable set of copies of p-points (the topological analogue of the Bernoulli scheme), and ω is the shift automorphism.

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Note: Figure translations are in progress. See original paper for figures.