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CONDITIONS FOR THE COMPLETE REGULARITY OF A CERTAIN CLASS OF RANDOM STATIONARY PROCESSES

(Presented by Academician Yu. V. Linnik, 23 X 1968)

A stationary random process $\xi(t)$, $-\infty < t < \infty$, is called completely regular if its coefficient of regularity

$$\rho(\tau) = \sup_{z_1, z_2} |E z_1 \overline{z}_2| \to 0, \qquad \tau \to \infty,$$

where the supremum is taken over all $z_1 \in H^0_{-\infty}$, $z_2 \in H^\infty_\tau$, $E|z_1|^2 = E|z_2|^2 = 1$. Here H^b_a denotes the closed, in the mean-square sense, linear span of the random variables $\xi(t)$, $a \le t \le b$.

Being regular, every completely regular process has a spectral density (s.d.) $f(\lambda)$, and by the theorem of M. G. Krein (see (1), p. 161)

$$\ln f/(1+\lambda^2) \in L^1(-\infty, \infty).$$

From the spectral decomposition of the process $\xi(t)$ it follows immediately that

$$\rho(\tau) = \rho(\tau; f) = \sup_{\varphi, \psi} \left| \int_{-\infty}^{\infty} e^{i\lambda\tau} \varphi(\lambda) \psi(\lambda) f(\lambda)\, d\lambda \right|, \tag{1}$$

where this time the supremum is taken over all φ, ψ belonging to the closure in the space L^2 with weight f of the system of functions $\{e^{it\lambda},\, t \ge 0\}$ and having there norms $\|\varphi\|_f = \|\psi\|_f = 1$.

The simplest class of stationary processes is the class of processes with s.d. equal to $|P(\lambda)|^{-2}$, where $P(\lambda)$ is a polynomial. In the case of Gaussian stationary processes, the processes of the indicated type exhaust all projections of Markov vector processes. It follows from (2) that all such processes are completely regular. A. M. Yaglom (3) showed that $\rho(\tau; |P|^{-2})$ is the maximal root of a certain deterministic equation. From his results there follows the equality: $\lim_{\tau\to\infty}(\rho(\tau))^{1/\tau} = e^{-\delta}$, where

$$\delta = \min |\operatorname{Im} z_j|$$

and the minimum is taken over all zeros z_j of the polynomial P.

Entire functions of finite degree form the next class of analytic functions in complexity, and one may think that in this situation it will still be possible to find a simple and definitive criterion for complete regularity. At the same time, the naturally arising problem of describing the set of entire functions of finite degree by which division preserves complete regularity would also be solved (4). Similar problems arise in the study of degenerate multidimensional completely regular processes (5).

Theorem. Let the s.d. $f(\lambda)$ of the stationary process $\xi(t)$ be equal to $1/g(\lambda)$, where $g(\lambda)$ is a nonnegative entire function of finite degree. The process $\xi(t)$ is completely regular if and only if:

1. $$\int_{-\infty}^{\infty} \frac{|\ln f(\lambda)|}{1+\lambda^2}\, d\lambda < \infty.$$

2. $$\sup_{-\infty<\lambda<\infty} \sum_j \left| \operatorname{Im} \frac{1}{\lambda - z_j} \right| < \infty.$$

Here the summation is over all zeros z_j of the function g. In addition, it is necessary that $\delta = \inf_j |\operatorname{Im} z_j| > 0$ and $\lim_\tau (\rho(\tau))^{1/\tau} = e^{-\delta}$.

Let us briefly outline the proof of the theorem. From the results of N. I. Akhiezer (6), on the basis of condition 1 it follows that $g(\lambda) = |\Gamma(\lambda)|^2$, where $\Gamma(z)$ is an entire function of finite degree, outer in the upper half-plane $\operatorname{Im} z > 0$, $z = \lambda + i\mu$. Relying on Lax' s theorem (7), it is not difficult to derive from (1) that

$$\rho(\tau) = \sup_\theta \left| \int_{-\infty}^{\infty} e^{i\tau\lambda} \theta(\lambda) \chi(\lambda)\, d\lambda \right|, \tag{2}$$

where $\chi(z) = \overline{\Gamma}(z)/\Gamma(z)$, $\overline{\Gamma}(z) = \overline{\Gamma(\bar{z})}$, and the supremum is taken over all θ from the unit sphere of H^1 in the upper half-plane. By virtue of condition 2, the meromorphic function $\chi(z)$ is analytic in the strip $|\operatorname{Im} z| < \delta$ and is bounded in any strip $|\operatorname{Im} z| \le \delta' < \delta$. Therefore, if $\Phi_r(\lambda)$ is an entire function of finite degree $\leq r$ realizing the best approximation to $\chi(\lambda)$, and $A_r(\chi)$ is the value of the best approximation of χ by entire functions of degree $\leq r$, then for all $\tau > r$

$$\rho(\tau) = \sup_\theta \left| \int_{-\infty}^{\infty} e^{i\tau\lambda} \theta(\lambda)[\chi(\lambda) - \Phi_r(\lambda)]\, d\lambda \right| \leq A_r(\chi) = O(e^{-r\delta}).$$

The necessity of condition 1 follows from Krein' s theorem already cited. Consequently, still $g = |\Gamma|^2$, and for $\rho(\tau)$ equality (2) holds. From the already proved condition 1 it follows [8] that $\sum |\operatorname{Im}(1/z_j)| < \infty$, and therefore the function $\chi(z)$ can be written in the form

$$\chi(z) = \alpha e^{i\beta z} \prod_j \left(1 - \frac{z}{z_j}\right)\left(1 - \frac{z}{\bar{z}_j}\right),$$

where $|\alpha| = 1$, and β is a real number. From this representation it is easy to infer that, for $\tau > |\beta|$, the function $e^{-i\lambda\tau}\overline{\chi} \in H^\infty$ in the upper half-plane. Therefore, for all $\theta \in H^1$ and all $\tau > |\beta|$,

$$\int_{-\infty}^{\infty} e^{-i\lambda\tau}\overline{\theta(\lambda)}\, \chi(\lambda)\, d\lambda = 0. \tag{3}$$

Lemma 1. Uniformly in λ as $t \to 0$,

$$\int_{\lambda-t}^{\lambda} \chi(s)\, ds - \int_{\lambda}^{\lambda+t} \chi(s)\, ds = o(t). \tag{4}$$

Relying on (2) and (3), with the aid of the methods used in [9], one can show that, whatever the three-times differentiable even function $a(\lambda)$, vanishing outside the interval $[-1, 1]$, uniformly in x as $T \to \infty$,

$$\frac{1}{T}\int_{-\infty}^{\infty} \frac{\sin^2 \frac{T\lambda}{2}}{\lambda^2}\, \chi(\lambda + x) a(T\lambda)\, d\lambda = o(1). \tag{5}$$

Since equality (5) has been proved for functions of the indicated kind, it is obvious that it will also remain valid for the odd function $a(\lambda)$, where $a(\lambda) = \lambda^2 \sin^{-2} \lambda/2,\ 0 < \lambda \leq 1,\ a(\lambda) = 0,\ \lambda > 1$, which is equivalent to (4).

Lemma 2. The function $\chi(z)$ is analytic in the strip $|\operatorname{Im} z| < \delta$, where $\delta > 0$.

Proof. Let $z_j = \alpha_j + i\beta_j$ be some zero of the function Γ. Put $\gamma_j = \left(1 - \dfrac{z}{z_j}\right)\Gamma$, $\overline{\gamma}_j = \left(1 - \dfrac{z}{\bar{z}_j}\right)\overline{\Gamma}$, and let $\varphi_j(\lambda) = \dfrac{1}{2\pi T_j}(e^{iT_j(\lambda-\alpha_j)} - 1)(\lambda - \alpha_j)^{-2}$, $T_j = |\beta_j|^{-1}$.

Putting in (2) $\theta = \varphi_j \dfrac{\overline{\gamma}_j}{\gamma_j}$ and evaluating the integral by residues, we find that $\rho(\tau) \geq \frac{1}{2} e^{\beta_j \tau}$, and, consequently, if $\delta = \inf |\beta_j|$, then $\rho(\tau) \geq \frac{1}{2} e^{-\delta\tau}$ and $\delta > 0$.

Lemma 3. The inequality $\sup\limits_{\lambda} |\chi'(\lambda)| < \infty$ holds.

Since

$$|\chi'(\lambda)| = \left|\beta - \sum \left|\operatorname{Im} \frac{1}{z_j - \lambda}\right|\right|,$$

the theorem will also be proved together with the lemma. Without loss of generality one may assume $\beta = 0$. Then

$$\chi^{(s+1)}(\lambda) = \sum_0^s C_s^k \chi^{(k)}(\lambda) \left(\sum_j \frac{2|\beta_j|}{(\alpha_j - \lambda)^2 + \beta_j^2} \right)^{(s-k)}.$$

The last equality, together with Lemma 2, makes it possible to prove that at all points λ where $|\chi'(\lambda)| \geq 1$ the inequality

$$|\chi^{(k)}(\lambda)| \leq L^k k! |\chi'(\lambda)|^k, \qquad k = 2, 3, \ldots \tag{6}$$

holds. The constant L does not depend on λ.

Let $\sup\limits_{\lambda} |\chi'(\lambda)| = \infty$. Choose points λ_k so that $M_k = |\chi'(\lambda_k)| \to \infty$, and put $t_k = (4M_{kL})^{-1}$.

By Lemma 2, for large k, in the domain $|\lambda - \lambda_k| \leq t_k$ the function $\chi(\lambda)$ is expanded in a Taylor series. Substituting this expansion into the left-hand side of (4) and using (6), we find

$$\left| \int_{\lambda_k - t_k}^{\lambda_k} \chi(\lambda)\, d\lambda - \int_{\lambda_k}^{\lambda_k + t} \chi(\lambda)\, d\lambda \right| \geq |\chi'(\lambda)| t_k^2 \left| 1 - 2 \sum_2^\infty (L t_k \chi'(\lambda_k))^s \right| \geq$$

$$\geq \frac{t_k^2}{3} |\chi'(\lambda_k)| = \frac{1}{12L} t_k \neq o(t_k),$$

which contradicts (4). The theorem is proved.

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Note: Figure translations are in progress. See original paper for figures.