ON THE METHOD OF INCOMPLETE FACTORIZATION

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 517.544

MATHEMATICS

Yu. I. CHERSKII

ON THE METHOD OF INCOMPLETE FACTORIZATION

(Presented by Academician N. I. Muskhelishvili on 21 III 1969)

In this note a special method is proposed for solving certain boundary-value problems of the theory of analytic functions; it consists of the following. Let a boundary condition be given on a contour L lying in the complex plane, and suppose that this condition contains as coefficients only such known functions as can be analytically continued from this contour and are piecewise holomorphic with jump lines along another contour Γ. The contours L and Γ do not coincide, but may have common points. Considering the given boundary condition as already partially factorized, one can then apply the usual solution procedure, known in the theory of the conjugation boundary-value problem ([1]). As a result of eliminating the discontinuities of the desired analytic functions along Γ, a **new boundary condition** arises on this contour. The new boundary-value problem on the contour Γ sometimes turns out to be simpler than the original one.

The method can be explained in more detail by the following examples.

Problem 1. Find a function $F(z)$, piecewise holomorphic (the jump line is the real axis) and bounded in the whole plane, from the boundary condition

$$\frac{1}{\sqrt{x^2+1}}\,F^+(x) + \lambda F^+(-x) - F^-(x) = G(x),$$

$$-\infty < x < \infty, \qquad \sqrt{x^2+1} > 0, \tag{1}$$

where $G(x)$ is a given function satisfying the condition H (Hölder condition) on the closed axis, and λ is a given real number different from zero.

Thus, in the present case the role of L is played by the real axis, and the contour Γ will be the cut issuing from the branch point $z = i$ of the square root $\sqrt{z^2+1}$. We also note that in problems of the class under consideration no conditions of analytic continuability need be imposed on the free terms, so that the contour Γ does not depend on the character of the function $G(x)$.

The solution of the problem consists of several stages.

1°. **Reduction of the boundary condition to a form convenient for analytic continuation.** Represent the known function $G(x)$ by the Sokhotski-Plemelj formula in the form of the difference $G^+(x) - G^-(x)$ of limiting values of analytic functions, and transfer to the right-hand side of condition (1) the functions analytically continuable into the lower half-plane:

$$F^+(x)/\sqrt{x^2+1} + \lambda F^+(-x) - G^+(x) = F^-(x) - G^-(x), \quad -\infty < x < \infty. \tag{2}$$

2°. **"Incomplete" analytic continuation.** On the right-hand side of the boundary condition (2) we have a function analytic in the lower half-plane, and on the left-hand side a function analytic in the upper half-plane, except for the cut issuing from the branch point $z = i$. In view of the condition $\sqrt{x^2+1} > 0$, this cut must go to infinity. Let us draw it along the imaginary axis, so that the contour Γ will be the ray $x = 0,\ y \geq 1$. Performing analytic continuation according to equality (2), we obtain—

we obtain a piecewise-holomorphic function with line of jumps Γ. Such a function can be represented by a Cauchy-type integral:

$$\frac{F^+(x)}{\sqrt{x^2+1}} + \lambda F^+(-x) - G^+(x) = F^-(x) - G^-(x) = I(x) + c, \tag{3}$$

where c is a constant, and

$$I(x) = \frac{1}{2\pi i}\int_i^{i\infty} \frac{p(\tau)\,d\tau}{\tau - x}.$$

The density $p(\tau)$ is an unknown function; to determine it an integral equation will be set up. Starting from equality (3) and taking into account the behavior of the Cauchy-type integral near the endpoint of the line of integration ((1), § 22), it is not difficult to find the class in which $p(\tau)$ should be sought. This is the class of functions defined on Γ, satisfying the Hölder condition at all interior points of this contour, and admitting, at its endpoints, representations

$$p(\tau) = O\left(\frac{1}{|\tau - i|^{1/2}}\right), \quad \tau \to i; \qquad p(\tau) = O\left(\frac{1}{|\tau|^{\mu}}\right), \quad |\tau| \to \infty,\ \mu > 0. \tag{4}$$

3°. **Expression of the unknown function $F^+(x)$ through the Cauchy-type integral $I(x)$:**

$$F^+(x) = \frac{x^2+1}{\lambda^2 x^2 + \lambda^2 - 1}\left\{\lambda\left[G^+(-x) + I(-x) + c\right] - \frac{G^+(x) + I(x) + c}{\sqrt{x^2+1}}\right\}. \tag{5}$$

4°. **Writing the continuity condition for the function $F^+(z)$ on the contour Γ.** By the condition of the problem, the function $F^+(z)$ must be analytic in the upper half-plane; therefore it is necessary to set

$$F^+(iy - 0) = F^+(iy + 0), \qquad y > 1. \tag{6}$$

Equality (6) can be given the form of a boundary condition on the contour Γ with the known piecewise-holomorphic function $I(z)$. We shall not write out this condition, but shall write the corresponding integral equation for the density p, which is easily obtained from (5) and (6), using the Sokhotskii–Plemelj formulas,

$$\frac{1}{\pi\sqrt{y^2-1}} \int_1^\infty \frac{p(i\xi)\,d\xi}{\xi - y} + \lambda p(iy) = -\frac{2iG^+(iy) + 2ic}{\sqrt{y^2-1}}, \qquad y > 1. \tag{7}$$

Here the integral is understood in the principal-value sense; $\sqrt{y^2-1} \geq 0$.

5°. **Solution of the obtained integral equation.** Put $p(i\xi) = u(\xi) + iv(\xi)$, where the functions $u(\xi)$ and $v(\xi)$ take real values. Since λ is real, equation (7) splits into two separate equations

$$\lambda u(y) + \frac{1}{\pi\sqrt{y^2-1}} \int_1^\infty \frac{u(\xi)\,d\xi}{\xi - y} = \frac{2}{\sqrt{y^2-1}} \operatorname{Im}[G^+(iy) + c], \qquad y > 1; \tag{8}$$

$$\lambda v(y) - \frac{1}{\pi\sqrt{y^2-1}} \int_1^\infty \frac{v(\xi)\,d\xi}{\xi - y} = \frac{2}{\sqrt{y^2-1}} \operatorname{Re}[G^+(iy) + c], \qquad y > 1. \tag{9}$$

Equations (8) and (9) are characteristic singular integral equations. The method for solving them in quadratures is known (see, for example, (1), § 97). By the substitution $\xi = 1/t$, $y = 1/x$ one can obtain characteristic equations on the interval $[0, 1]$, whose solutions, in accordance with (4), should be sought in the class H^*. Having determined the functions $u(\xi)$ and $v(\xi)$, we find the density $p(\tau)$, and with it the solution of the posed problem.

It may seem that the requirement of "piecewise-holomorphic" continuity of the coefficients of the boundary condition substantially narrows the range of applicability of the method under consideration. It should be noted, however, that in applications the coefficients of boundary-value problems turn out precisely to be "piecewise-holomorphically" continuable. Examples are functions such as $|x|$, $\operatorname{sgn} x$, radicals of rational functions, etc. The following conjugation problem for a system of two pairs of functions leads to the fundamental mixed problem for an elastic half-space, as well as to some other problems of mathematical physics (see ([2]), p. 218).

Problem 2. Find two piecewise-holomorphic functions $\Phi(z)$ and $\Psi(z)$ (the line of jumps is the real axis), bounded in the whole plane, satisfying the boundary conditions

$$\Phi^+(x) = \sqrt{x^2+1}\,\Phi^-(x) + \Psi^-(x) + G(x)), \qquad \sqrt{x^2+1}\Psi^+(x) = \sqrt{x^2+1}$$

$$\Phi^-(x) - \delta\Psi^-(x) + H(x), \qquad -\infty < x < \infty, \tag{10}$$

where $G(x)$ and $H(x)$ are given functions satisfying the condition H on the compactified axis, and λ is a given number, $\lambda \neq -1$.

The method of incomplete factorization makes it possible to solve the posed problem in quadratures. The solution is carried out according to the stages indicated above.

$$1,\,2. \quad \Phi^+(x) - \sqrt{x^2+1}\Psi^+(x) - G^+(x) + H^+(x) = \frac{1}{2\pi i}\int_i^{i\infty} \frac{p(\tau)\,d\tau}{\tau - x} + c_p,$$

$$\Phi^+(x) - G^+(x) = \sqrt{x^2+1}\Phi^-(x) + \Psi^-(x) - G^-(x) = \frac{1}{2\pi i}\int_{-i\infty}^{-i} \frac{q(\tau)\,d\tau}{\tau - x} + c_q$$

(c_p and c_q are constants).

$$3. \quad \Psi^+(x) = -\frac{1}{\sqrt{x^2+1}}\left\{\frac{1}{2\pi i}\int_i^{i\infty}\frac{p(\tau)\,d\tau}{\tau - x} + c_p - \Phi^+(x) + G^+(x) - H^+(x)\right\}$$

$$\Phi^-(x) = \frac{1}{\sqrt{x^2+1}}\left\{\frac{1}{2\pi i}\int_{-i\infty}^{-i}\frac{q(\tau)\,d\tau}{\tau - x} + c_q - \Psi^-(x) + G^-(x)\right\}.$$

4. We find the limiting values of the function $\Psi^+(z)$ for $z = iy,\ y > 1:$

$$\Psi^+(iy{-}0) = -\frac{i}{\sqrt{y^2-1}}\left\{\frac{p(iy)}{2} + \frac{1}{2\pi i}\int_i^{i\infty}\frac{p(\tau)\,d\tau}{\tau - iy} + c_p - \Phi^+(iy) + G^+(iy) - H^+(iy)\right\},$$

$$\Psi^+(iy+0) =$$

$$= \frac{i}{\sqrt{y^2-1}} \left\{ -\frac{p(iy)}{2} + \frac{1}{2\pi i} \int_i^{i\infty} \frac{p(\tau)\,d\tau}{\tau - iy} + c_p - \Phi^+(iy) + G^+(iy) - H^+(iy) \right\},$$

and also the limiting values of the function $\Phi^-(z)$ for $z = iy,\ y < -1$

$$\Phi^-(iy-0) = \frac{i}{\sqrt{y^2-1}} \left\{ \frac{q(iy)}{2} + \frac{1}{2\pi i} \int_{-i\infty}^{-i} \frac{q(\tau)\,d\tau}{\tau - iy} + c_q - \Psi^-(iy) + G^-(iy) \right\},$$

$$\Phi^-(iy+0) = -\frac{i}{\sqrt{y^2-1}} \left\{ -\frac{q(iy)}{2} + \frac{1}{2\pi i} \int_{-i\infty}^{-i} \frac{q(\tau)\,d\tau}{\tau - iy} + c_q - \Psi^-(iy) + G^-(iy) \right\}.$$

Conditions analogous to equality (6) have the form

$$\Psi^+(iy-0) = \Psi^+(iy+0), \quad y > 1; \qquad \Phi^-(iy-0) = \Phi^-(iy+0), \quad y < -1.$$

After simple transformations these conditions take the form of a system of integral equations

$$\frac{1}{2\pi i} \int_i^{i\infty} \frac{p(\tau)\,d\tau}{\tau - iy} - \frac{1}{2\pi i} \int_{-i\infty}^{-i} \frac{q(\tau)\,d\tau}{\tau - iy} = H^+(iy) + c_q - c_p, \qquad y > 1, \tag{11}$$

$$\frac{1}{2\pi i} \int_i^{i\infty} \frac{p(\tau)\,d\tau}{\tau - iy} - \frac{1+\lambda}{2\pi i} \int_{-i\infty}^{-i} \frac{q(\tau)\,d\tau}{\tau - iy} = \lambda G^-(iy) + H^-(iy) + (1+\lambda)c_q - c_p,$$

$$y < -1.$$

5. To reduce system (11) to Wiener-Hopf equations, introduce the notation: $f_1(t) = p(\tau)$, where $\tau = ie^t,\ 0 < t < \infty$; $f_2(t) = q(\tau)$, where $\tau = -ie^t,\ 0 < t < \infty$. We obtain a system of the form

$$\frac{1}{2\pi i} \int_0^{\infty} \frac{f_1(t)\,dt}{1 - e^{x-t}} + \frac{1}{2\pi i} \int_0^{\infty} \frac{f_2(t)\,dt}{1 + e^{x-t}} = H^+(ie^x) + c_q - c_p, \qquad x > 0; \tag{12}$$

$$\frac{1}{2\pi i} \int_0^{\infty} \frac{f_1(t)\,dt}{1 + e^{x-t}} + \frac{1+\lambda}{2\pi i} \int_0^{\infty} \frac{f_2(t)\,dt}{1 - e^{x-t}} = \lambda G^-(-ie^x) + H^-(-ie^x) +$$

$$+(1+\lambda)c_q - c_p, \qquad x > 0. \tag{13}$$

Multiply (12) by a fixed value of the root $\sqrt{1+\lambda}$. Adding and subtracting (12) and (13), we obtain two independent equations

$$\frac{1}{2\pi i}\int_0^\infty \left[\frac{\sqrt{1+\lambda}}{1-e^{x-t}} + \frac{1}{1+e^{x-t}}\right]\varphi(t)\,dt = h_1(x) +$$

$$+(\sqrt{1+\lambda}+1+\lambda)c_q - (\sqrt{1+\lambda}+1)c_p, \qquad x > 0; \tag{14}$$

$$\frac{1}{2\pi i}\int_0^\infty \left[\frac{\sqrt{1+\lambda}}{1-e^{x-t}} - \frac{1}{1+e^{x-t}}\right]\psi(t)\,dt = h_2(x) +$$

$$+(\sqrt{1+\lambda}-1-\lambda)c_q - (\sqrt{1+\lambda}-1)c_p, \qquad x > 0, \tag{15}$$

where $\varphi(t) = f_1(t) + \sqrt{1+\lambda}\, f_2(t)$, $\psi(t) = f_1(t) - \sqrt{1+\lambda}\, f_2(t)$, and $h_1(x)$ and $h_2(x)$ are known.

It is known that, by means of the Fourier transform, equations (14) and (15) reduce to conjugation problems. The details of the solution of these equations are beyond the scope of this note.

Odessa State University
named after I. I. Mechnikov

Received
10 III 1969

REFERENCES

1. N. I. Muskhelishvili, *Singular Integral Equations*, Moscow, 1962.

2. Yu. I. Cherskii, *Tr. Tbilissk. matem. inst.*, 26 (1962).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.