AN A PRIORI ESTIMATE OF THE PRINCIPAL RADII OF CURVATURE OF A CLOSED CONVEX HYPERSURFACE IN TERMS OF THEIR MEAN VALUES

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Abstract

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AN A PRIORI ESTIMATE OF THE PRINCIPAL RADII OF CURVATURE OF A CLOSED CONVEX HYPERSURFACE IN TERMS OF THEIR MEAN VALUES

We consider a closed convex hypersurface F in $(n+1)$-dimensional Euclidean space, satisfying the equation

$$S_k(R_1, \dots, R_n) = \overline{S}_k(\nu), \tag{1}$$

where S_k is the elementary symmetric function of the principal radii of curvature of the hypersurface at a point with exterior normal ν, and $\overline{S}_k(\nu)$ is a prescribed function of the unit vector ν. The problem is to estimate the principal radii of curvature of the hypersurface in terms of the function $\overline{S}_k$. In the case $k = n$ such an estimate was obtained in the author' s paper (^{1}).

Denote

$$\overline{\rho}_k(\nu) = \left(\frac{\overline{S}_k(\nu)}{C_n^k}\right)^{1/k}$$

and call the quantity $\overline{\rho}_k(\nu)$ the mean radius of normal curvature of the hypersurface F at the point with exterior normal ν.

Theorem. *For the radii of normal curvature of a closed convex hypersurface F the estimate*

$$R \le \max_{X,\gamma}(\overline{\rho}_k(X) - \overline{\rho}_k''(X)),$$

holds, where the differentiation of $\overline{\rho}_k$ is performed along the arc of the great circle γ issuing from the point X on the unit sphere Ω, and the maximum is taken over all points X of the sphere and over all directions γ from this point.

Proof. Let H be the support function of the hypersurface F. Then the principal radii of curvature R_k are the roots of the polynomial

$$P(R) = \|H_{ij} + R\delta_{ij}\| = 0,$$

where H_{ij} are the second derivatives of H on the unit sphere. Put

$$h(x_1, x_2, \dots, x_n) = H(x_1, x_2, \dots, x_n, 1).$$

Then, taking into account the homogeneity of the function H, the derivatives H_{ij} can be expressed in terms of the derivatives of the function h, and the equation $P(R) = 0$ for the principal radii of curvature, after elementary transformations, is represented in the form

$$P(R) = \begin{vmatrix} \lambda h_{11} + R & \lambda h_{12} & \cdots & x_1 R \\ \lambda h_{21} & \lambda h_{22} + R & \cdots & x_2 R \\ \cdots & \cdots & \cdots & \cdots \\ x_1 R & x_2 R & \cdots & R(1 + x_1^2 + \cdots + x_n^2) \end{vmatrix} = 0, \tag{2}$$

where

$$h_{ij} = \partial^2 h / \partial x_i \partial x_j, \qquad \lambda = (1 + x_1^2 + \cdots + x_n^2)^{1/2}, \tag{3}$$

$$P(R) = R^{n+1} + S_1 R^n + \cdots + S_{nR}.$$

The coefficients S_k of the polynomial $P(R)$ are elementary symmetric functions of the principal radii of curvature,

$$S_k(R_1 \dots R_n) = \sum_{i_\alpha \neq i_\beta} R_{i_1} R_{i_2} \dots R_{i_k}.$$

As was shown in (^{1}), one can introduce a coordinate system $x_1, \dots, x_n$ such that the function

$$w = h_{11}(1 + x_1^2 + \cdots + x_n^2)^{3/2} / (1 + x_2^2 + \cdots + x_n^2)$$

at the point $O : x_1 = x_2 = \cdots = x_n = 0$ attains an absolute maximum, and this maximum is equal to the largest radius of normal curvature of the hypersurface F. Thus the problem reduces to estimating this maximum. At the point O

$$h_{ii} = R_i, \quad h_{ij} = 0 \text{ for } i \neq j; \quad w_i = h_{11i} = 0; \tag{4}$$

$$w_{11} = (h_{11})_{11} + 3R_1 \leqslant 0, \quad w_{ii} = (h_{11})_{ii} + R_1 \leqslant 0, \quad i > 1. \tag{5}$$

Let us compute the derivative $S'_k = dS_k/dx_1$ at the point O. It is the coefficient of the polynomial $P' = dP/dx_1$. We have

$$P' = \sum_i R(R_1 + R)\dots(R_{i-1} + R)(R_{i+1} + R)\dots(R_n + R)h'_{ii}.$$

Here the summation is over $i > 1$. But since $h'_{11} = 0$, it may be assumed that the summation extends over all values of i from 1 to n. The coefficient of R^{n-k} in the polynomial P' has the form

$$S'_{k+1} = \sum_i h'_{ii} S^i_k,$$

where S^i_k is the elementary symmetric function of the variables $R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n$. Noting that $S^i_k = \partial S_{k+1}/\partial R_i$, we obtain

$$S'_{k+1} = \sum_i h'_{ii} \frac{\partial S_{k+1}}{\partial R_i}.$$

Thus,

$$S'_k = dS_k, \quad \text{if } dR_i = h'_{ii}. \tag{6}$$

We now compute the second derivative S''_k with respect to the variable x_1 at the point O. For this we differentiate the polynomial P with respect to x_1 twice. Omitting the corresponding calculations, we give the value of the coefficient S''_k that interests us:

$$S''_k = S^i_{k-1}(h_{11})_{ii} + (k+2)S'_k - 2S'_k + 2S^{ij}_{k-2}(h'_{ii}h'_{jj} - h'^{2}_{ij}),$$

where S^{ij}_{k-2} is the elementary symmetric function of the variables $R_1, R_2, \dots,$ except for R_i and R_j. For simplicity of notation, the summation sign will henceforth be omitted.

Taking into account the inequalities (5), after simple transformations we obtain

$$S''_k \leqslant -R_1(n-k+1)S_{k-1} + kS_k + 2S^{ij}_{k-2}(h'_{ii}h'_{jj} - h'^{2}_{ij}).$$

We strengthen this inequality by dropping the term $-h'^{2}_{ij}$. Then

$$S_k'' \leqslant -R_1(n-k+1)S_{k-1} + kS_k + 2S^{ij}_{k-2}h'_{ii}h'_{jj}, \tag{7}$$

Note that

$$S^{ij}_{k-2}h'_{ii}h'_{jj} = d^2 S_k, \quad \text{if } dR_i = h'_{ii},$$

and use the concavity of the function $(S_k)^{1/k}$ in the variables R_i (^{2}). We have

$$d^2(S_k^{1/k}) \leqslant 0.$$

Hence

$$d^2 S_k \leq \left(1 - \frac{1}{k}\right) \frac{(dS_k)^2}{S_k} = \left(1 - \frac{1}{k}\right) \frac{(S_k')^2}{S_k}.$$

Now inequality (7) is strengthened as follows:

$$S_k'' \leq -R_1(n-k+1)S_{k-1} + kS_k + \left(1 - \frac{1}{k}\right) \frac{S_{k'}^2}{S_k}.$$

Hence

$$R_1 \leq \frac{kS_k + (1-1/k)S_{k'}^2/S_k - S_k''}{(n-k+1)S_{k-1}}. \tag{8}$$

Noting that

$$\left(\frac{S_{k-1}}{C_n^{k-1}}\right)^{1/(k-1)} \geq \left(\frac{S_k}{C_n^k}\right)^{1/k},$$

from (8) we finally obtain the required estimate

$$R \leq \left(\frac{S_k}{C_n^k}\right)^{1/k} - \left[\left(\frac{S_k}{C_n^k}\right)^{1/k}\right]''.$$

Differentiation with respect to x_1 at the point O may be replaced by differentiation along the arc of a great circle issuing in the direction of the axis x_1. The theorem is proved.

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2. E. Beckenbach, R. Bellman, *Inequalities*, Moscow, 1965.

Note: Figure translations are in progress. See original paper for figures.