ON A SYSTEM OF BELTRAMI EQUATIONS DEGENERATING ON A LINE

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Abstract

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MATHEMATICS

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ON A SYSTEM OF BELTRAMI EQUATIONS DEGENERATING ON A LINE

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1. Consider the equation

$$\partial w/\partial \bar{z} - q(x)\partial w/\partial z = 0, \tag{1}$$

where $z = x + iy$, $\partial/\partial\bar{z} = \frac{1}{2}(\partial/\partial x + i\partial/\partial y)$, $\partial/\partial z = \frac{1}{2}(\partial/\partial x - i\partial/\partial y)$, $q(z)$ is a given, and $w(z)$ the sought, complex-valued function. This equation is equivalent to a system of two real first-order equations, known as the system of Beltrami equations in the case of its uniform ellipticity; moreover, the condition of uniform ellipticity for (1) has the form

$$|q(z)| \leqslant \text{const} < 1. \tag{2}$$

In the present note we shall consider equation (1) when condition (2) is violated along a certain line. Let $|q(z)|$ be an analytic function of the variables x, y in some domain D, and let the set of points of the domain D at which $|q(z)| = 1$ ($q(z) \neq 1$) be a simple analytic curve γ.

If, along γ, all partial derivatives of the function $|q(z)|^2 - 1$ up to order $n-1$ (inclusive) are equal to zero, while among the derivatives of order n at least one is different from zero, then in some neighborhood $\Delta \subset D$ of an arc of γ there is the representation

$$|q(z)|^2 - 1 = \eta^n(x, y)a(x, y), \tag{3}$$

where $\eta(x, y) = 0$ is the equation of the curve γ, and $a(x, y) \neq 0$; moreover the functions $\partial\xi/\partial x$ and $\partial\xi/\partial y$ are not simultaneously equal to zero in Δ.

2. Suppose that the direction of the characteristics of equation (1) at the points of the arc γ does not coincide with the direction of the tangent to the arc γ, i.e. that along γ the inequality

$$(1 - \operatorname{Re} q(z))\partial\eta/\partial x - \operatorname{Im} q(z)\partial\eta/\partial y \neq 0 \tag{4}$$

holds.

Let the function $\xi(x, y)$ be a solution of the equation

$$(1 - \operatorname{Re} q(z))\partial\xi/\partial x - \operatorname{Im} q(z)\partial\xi/\partial y = 0. \tag{5}$$

It is evident that one can find a subdomain δ of the domain Δ, containing inside it an arc of γ, in which the function $b(x, y)$, satisfying the equalities

$$\partial\xi/\partial x = b(x, y) \operatorname{Im} q(z), \qquad \partial\xi/\partial y = b(x, y)(1 - \operatorname{Re} q(z)), \tag{6}$$

is different from zero. Since

$$I = \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial y} = -b(x, y)\{(1 - \operatorname{Re} q(z))\partial\eta/\partial x - \operatorname{Im} q(z)\partial\eta/\partial y\} \neq 0$$

in δ, the mapping $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ is a homeomorphism of the neighborhood δ onto some domain $\tilde{\delta}$ in the plane of the variables ξ, η. As a result of this mapping, equation (1) is transformed to the form:

$$(\partial w/\partial\bar{\zeta} + \partial w/\partial\zeta)(\partial\xi/\partial\bar{z} - q(z)\partial\xi/\partial z) -$$

$$-i(\partial w/\partial\bar{\zeta} - \partial w/\partial\zeta)(\partial\eta/\partial\bar{z} - q(z)\partial\eta/\partial z) = 0, \tag{7}$$

where $\zeta = \xi + i\eta$, $\partial/\partial\bar{\zeta} = \frac{1}{2}(\partial/\partial\xi + i\partial/\partial\eta)$, $\partial/\partial\zeta = \frac{1}{2}(\partial/\partial\xi - i\partial/\partial\eta)$.

If we now take into account that, by virtue of equalities (6) and (3),

$$\frac{\partial\xi}{\partial\bar{z}} - q(z)\frac{\partial\xi}{\partial z} = -\frac{ib(x, y)}{2}\left(|q(z)|^2 - 1\right) = -\frac{ia(x, y)b(x, y)}{2}\eta^n,$$

and, by virtue of inequality (4), $\partial\eta/\partial\bar{z} - q(z)\partial\eta/\partial z \neq 0$ in δ, then, dividing (7) by $\eta^*(z)$, we shall have

$$(1 + \eta^n c(\xi))\partial w/\partial\bar{\xi} - (1 - \eta^n c(\xi))\partial w/\partial\xi = 0, \tag{8}$$

where

$$c(\xi) = a[x(\xi, \eta), y(\xi, \eta)]b[x(\xi, \eta), y(\xi, \eta)]/\eta^*[z(\xi)], \quad \eta^*(x) = \partial\eta/\partial\bar{z} - q(z)\partial\eta/\partial z.$$

Let us note that, by virtue of inequality (4), the function $\operatorname{Re} c(\xi)$ is different from zero in the domain δ, since

$$\operatorname{Re} c(\xi) = \frac{ab}{|\eta^*|^2}\left\{(1-\operatorname{Re} q)\frac{\partial\eta}{\partial x} - \operatorname{Im} q\cdot\frac{\partial\eta}{\partial y}\right\} \neq 0 \quad \text{in } \delta.$$

3. Let us now consider the case when along γ the equality

$$(1-\operatorname{Re} q(z))\partial\eta/\partial x - \operatorname{Im} q(z)\partial\eta/\partial y = 0 \tag{9}$$

holds. In this case one can find functions $\mu(x, y)$ and $\nu(x, y)$ for which the inequality

$$(1-\operatorname{Re} q(z))\mu(x, y) - \operatorname{Im} q(z)\cdot\nu(x, y) \neq 0 \tag{10}$$

holds.

Let $\xi(x, y)$ be a solution of equation (5), and let $\tilde\eta(x, y)$ satisfy the equality

$$\nu(x, y)\partial\tilde\eta/\partial x - \mu(x, y)\partial\tilde\eta/\partial y = 0. \tag{11}$$

Obviously, one can find such an arc γ_1 of the arc γ and a domain δ, containing within itself the arc γ_1, in which inequality (10) holds and the functions $b_1(x, y)$, $b_2(x, y)$, distinct from zero, satisfy the relations

$$\begin{aligned} \partial\xi/\partial x &= b_1(x, y)\operatorname{Im} q(z), \quad & \partial\xi/\partial y &= b_1(x, y)(1-\operatorname{Re} q(z)),\\ \partial\tilde\eta/\partial x &= b_2(x, y)\mu(x, y), \quad & \partial\tilde\eta/\partial y &= b_2(x, y)\nu(x, y). \end{aligned} \tag{12}$$

From these relations and inequalities (11) it follows that the Jacobian of the transformation $\xi = \xi(x, y)$, $\tilde\eta = \tilde\eta(x, y)$ is different from zero in the neighborhood δ:

$$J = b_1 b_2(\operatorname{Im} q(z)\cdot\nu(x, y) - (1-\operatorname{Re} q(z))\mu(x, y)) \neq 0.$$

Let $\eta^0(\xi, \tilde\eta) = \eta[x(\xi, \tilde\eta), y(\xi, \tilde\eta)]$. Since along γ_1 the directions of the tangent curves $\eta(x, y) = \text{const}$, $\xi(x, y) = \text{const}$, by virtue of (5) and (9), coincide, it follows that $\eta^0(0, \tilde\eta) = 0$. But it is easy to see that

$$\frac{\partial\eta^0}{\partial\xi} = \frac{b_2}{J}\left(\nu(x, y)\frac{\partial\eta}{\partial x} - \mu(x, y)\frac{\partial\eta}{\partial y}\right) \neq 0.$$

Therefore, in some neighborhood $\delta_1 \subset \delta$ of the arc γ_1, the function $\eta^0(\xi, \tilde\eta)$ has the form

$$\eta^0(\xi,\tilde\eta)=\xi\eta_0(\xi,\tilde\eta),\qquad \eta_0(\xi,\tilde\eta)\neq 0,\tag{13}$$

and since

$$\partial\xi/\partial\bar z-q(z)\partial\xi/\partial z=-{}^1/{}_2 ib_1\left(|q(z)|^2-1\right)=[\eta^0(\xi,\tilde\eta)]^n a[x(\xi,\eta),y(\xi,\eta)],$$

then, taking (13) into account, we are convinced that, as a result of the non-degenerate transformation $\xi=\xi(x,y)$, $\tilde\eta=\tilde\eta(x,y)$, equation (1) takes the form ($\zeta=\xi+i\tilde\eta$)

$$(1+\xi^n c_0(\zeta))\partial w/\partial\bar\zeta-(1-\xi^n c_0(\zeta))\partial w/\partial\zeta=0,\tag{14}$$

where

$$c_0(\zeta)=a_1b_1\eta_0(\xi,\tilde\eta)/2(\partial\tilde\eta/\partial\bar z-q(z)\partial\tilde\eta/\partial z)_{z=z(\zeta)}.$$

By virtue of inequality (10) it is easy to see that the function $\operatorname{Re}c_0(\zeta)$ is different from zero in the domain $\tilde\delta_1$—the image of the domain δ_1 under the mapping $\xi=\xi(x,y)$, $\eta=\eta(x,y)$. Thus, in a neighborhood of the line of degeneration γ, equation (1) can always be reduced either to the form (8) or to the form (14).

4. Let G be a domain in the (ξ,η)-plane, bounded by some arc σ, situated in the half-plane $\eta>0$ (or $\eta<0$) and adjacent to the axis $\eta=0$, and also by a segment AB of the real axis $\eta=0$. In the domain G consider an equation of the form (8), and we shall assume that $c(\zeta)$ is a function of class C'_α in the domain $G+\sigma+AB$ and that $\operatorname{Re}c(\zeta)>0$ (if n is an even number, then, obviously, the latter condition does not restrict generality). Let $w(\zeta)$ be a solution of equation (8), regular in the domain G, continuous in $G+\sigma+AB$. Making the change of variable

$$\zeta_*(\zeta)=\xi_*+i\eta_*=\xi+\frac{i}{n+1}\eta^{n+1},$$

which maps the domain G onto the domain G_*, situated in the half-plane $\eta^*>0$ (or $\eta^*<0$), bounded by the arc σ and the segment AB of the axis $\eta^*=0$, instead of equation (8) we obtain the Beltrami equation

$$\frac{\partial w}{\partial\overline{\zeta_*}}-\frac{1-c(\zeta_*)}{1+c(\zeta_*)}\frac{\partial w}{\partial\zeta_*}=0.\tag{15}$$

Since $\operatorname{Re}c(\zeta)>0$ in $\overline{G}$, the function $\operatorname{Re}c(\zeta_*)$ is positive in the domain $\overline{G_*}$, and, consequently, the modulus of the function $(1-c(\zeta_*))/(1+c(\zeta_*))$ is bounded in

G_* from above by unity; in this case, as is known ([1]), all regular solutions of equation (15) have the form

$$w = \Phi[\chi(\zeta_*)], \tag{16}$$

where $\chi(\zeta_*)$ is a univalent solution of equation (15), and $\Phi(\chi)$ is an arbitrary holomorphic function. Consequently, all solutions of equation (8) regular in G, continuous in the domain $G+\sigma+AB$, are represented by the formula

$$w(\zeta) = \Phi\left[\chi\left(\xi + \frac{i}{n+1}\eta^{n+1}\right)\right]. \tag{17}$$

If one uses the known representations for holomorphic functions in the form of Cauchy-type integrals with real density, then, using formula (17), it is easy to study the following boundary-value problem: find regular solutions of equation (8), continuous in $G+\sigma+AB$, and satisfying the condition

$$\mathrm{Re}[\lambda(\zeta)w(\zeta)] = h(\zeta), \qquad \zeta \in \sigma + AB.$$

Let $\varkappa$ be the integer equal to the increment of the function $\arg\lambda(\zeta)$, obtained when the contour $\sigma - AB$ is traversed once counterclockwise, divided by 2π. Then, with respect to this problem, the following assertions hold: if $\varkappa \geq 0$, then the homogeneous problem has exactly $2\varkappa+1$ linearly independent solutions, and the nonhomogeneous problem is solvable; but if $\varkappa < 0$, then the homogeneous problem has no nontrivial solution, and the nonhomogeneous problem is solvable provided that $-2\varkappa-1$ conditions are satisfied by the right-hand side $h(\zeta)$. In particular, the Dirichlet problem

$$\mathrm{Re}\, w(\zeta) = h(\zeta), \qquad \zeta \in \sigma + AB$$

for equation (8) is always solvable uniquely up to an imaginary constant. An analogous problem can also be considered for equations of the form (14).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.