ON CONDITIONS FOR THE EXISTENCE OF BOUNDED SOLUTIONS OF DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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Abstract

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ON CONDITIONS FOR THE EXISTENCE OF BOUNDED SOLUTIONS OF DIFFERENTIAL EQUATIONS IN A BANACH SPACE

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Consider in a Banach space E the differential equation

$$dx/dt = A(t)x + f(t), \tag{1}$$

where $x = x(t)$ is the sought function with values in E; $A(t)$ is, generally speaking, for each fixed $t \in (-\infty, \infty)$ an unbounded linear operator with domain D, dense in E and independent of t; $f(t)$ is a bounded strongly continuous function on $(-\infty, \infty)$,

$$\sup_t \|f(t)\| \leq N.$$

There are various conditions for the existence of a bounded solution of this equation. For example, in [1] the case is indicated when equation (1) has at least one generalized solution bounded on $[0, \infty)$, and in [2] the Galerkin method is used to establish the existence of a bounded solution on the entire number axis.

In the present paper, by a somewhat different route, a method is indicated for finding a bounded solution of equation (1). Here the method of approximating a bounded solution of equation (1) by a bounded solution of the corresponding difference equation is essentially used:

$$x(t) = (I + hA(t))^{-1}(x(t+h) - hf(t)). \tag{2}$$

In what follows it is assumed that the operator $A(t)$ has a bounded inverse operator, and also that all the operators $A(t)A^{-1}(s)$ are bounded and satisfy the condition

$$\|I - A(t)A^{-1}(s)\| \leq M|t-s|, \qquad t, s \in (-\infty, \infty), \tag{3}$$

where I is the identity operator; M is some positive number. In addition, it is assumed that for all $h \in (0, H)$ there exist bounded linear operators $(I + hA(t))^{-1}$ with the estimate

$$\|(I + hA(t))^{-1}\| \leq 1 - ph, \qquad t \in (-\infty, \infty), \tag{4}$$

where p is some positive number, and H is a sufficiently small positive number.

Lemma 1. *Let conditions (3) and (4) be fulfilled. Then equation (2), for all $h \in (0, H)$, has one and only one continuous bounded solution $x(t, h)$ on $(-\infty, \infty)$; moreover, the estimate*

$$\sup_{t,h} \|x(t, h)\| \leq N/p$$

holds.

The following proposition establishes the relation between the bounded solutions of equations (1) and (2).

Lemma 2. *Let $A(t)$ be a bounded operator, defined for all t on all of E and satisfying conditions (3) and (4). Then, if from the set of solutions $x(t, h_m)$ of equation (2), where $h_m \to 0$ as $m \to \infty$, one can select a subsequence converging uniformly on each finite interval to some continuous function $x(t)$, then $x(t)$ is*

is called a bounded solution of equation (1), and if $x(t)$ is a bounded solution of equation (1), then

$$\lim_{h \to 0} x(t, h) = x(t).$$

Let

$$A_n(t) = A(t) \left(I + \frac{1}{n} A(t)\right)^{-1}$$

and let n take integer values greater than $2H^{-1}$. Then the operators $A_n(t)$ for all $h \in (0, H/2)$ satisfy conditions (3) and (4). Moreover, the constants p and M are the same and do not depend on n. Bounded solutions of equation (1) can be approximated by bounded solutions of the equation

$$dx/dt = A_n(t)x + f(t). \tag{5}$$

Lemma 3. Let the operator $A^2(t)$ have a domain of definition $D(A^2(t))$ independent of t. Let $f(t)$, for each t, take values in $D(A^2(t))$ and satisfy the estimate $\sup \|A^2(t)f(t)\| < +\infty$. In addition, let all operators $A^2(t)A^{-2}(s)$ be bounded and let the inequality

$$\|I - A^2(t)A^{-2}(s)\| \leq M_1|t-s|, \qquad t,s \in (-\infty,\infty), \tag{6}$$

hold, where M_1 is some constant number. If conditions (3) and (4) hold and the inequality $\max\{M, M_1\} < p$ holds, then the sequence of bounded solutions $x_n(t)$ of the difference equation corresponding to equation (5) converges uniformly in h to the function $x(t,h)$.

In what follows, a continuously differentiable function $x(t)$, bounded on $(-\infty,\infty)$ and satisfying equation (1), will be called a classical solution of this equation. The uniqueness of the classical bounded solution follows from the next lemma.

Lemma 4. Let the equation

$$dx/dt = A_n(t)x + g_n(t),$$

where $g_n(t)$ is some continuous, generally speaking unbounded on $(-\infty,\infty)$, function, have a continuous solution $x_n(t)$ uniformly bounded in n on the whole number axis. If the function $g_n(t)$ tends to zero as $n \to \infty$, uniformly in t on every finite interval, then for any fixed t the function $x_n(t)$ tends to zero as $n \to \infty$.

For the proof, for $\varepsilon > 0$ on some interval one constructs a set of points τ_{nh} such that

$$\|x_n(t)\| \leq \|x_n(\tau_{nh})\|, \qquad \|x_n(\tau_{nh}+h)\| - \|x_n(\tau_{nh})\| \leq \varepsilon h.$$

Then from the identity

$$x_n(t+h) = x_n(t) + hA_n(t)x_n(t) + hg_n(t) + \alpha(t, x_n(t), h), \tag{7}$$

where the quantity $\alpha(t, x_n(t), h)$, for fixed n, tends to zero together with h uniformly in t on every finite interval, one can obtain the estimate

$$\|x_n(\tau_{nh})\| \leq \frac{1}{p}\left(\varepsilon + \|g_n(\tau_{nh})\| + \|\alpha(\tau_{nh}, x_n, h)\|\right),$$

from which the lemma follows. Incidentally, the same idea is used in the proof of Lemmas 2 and 3.

Lemma 5. Let $x(t)$ be a classical bounded solution of equation (1), and let $x_n(t)$ be a continuous solution of equation (5), uniformly bounded in n on $(-\infty,\infty)$. Then

$$\lim_{n\to\infty} x_n(t) = x(t).$$

The proof of this lemma is based on Lemma 4.

Definition 1. If from the set of functions $x_n(t, h_m)$, where $h_m \to 0$ as $m \to \infty$, one can select a subsequence converging to some function $x_n(t)$, and from the obtained set of functions $x_n(t)$, in turn, a subsequence converging to the function $x^*(t)$, then

the function $x^*(t)$ is called a **generalized bounded solution** of equation (1).

This definition is justified by the fact that, if equation (1) has a classical bounded solution, then, by Lemma 5, the generalized solution coincides with the classical one.

Theorem 1. *Suppose that conditions (3) and (4) are satisfied. Suppose that $f(t)$, for every t, takes values in D and satisfies the condition*

$$\sup \|A(t)f(t)\| < +\infty.$$

Then, if $f(t)$ is a uniformly continuous function on $(-\infty, \infty)$, $A^{-1}(t)$ is a completely continuous operator for each fixed t, and $p > M$, equation (1) has at least one bounded generalized solution. If, in addition, the operator $A^2(t)$ has a domain of definition independent of t, satisfies condition (6), and $p > M_1$, then equation (1) has a unique bounded generalized solution.

Proof. Since the functions $A(t)x(t, h)$ and $A(t)x_n(t, h)$ exist and are bounded uniformly in t, n, and h, the values of the functions $x(t, h)$ and $x_n(t, h)$ for each t form compact sets in E. Then, from their equicontinuity on the entire real axis, by Lemmas 1 and 2, the existence of a generalized solution follows. Further, by Lemmas 2 and 3, the equation

$$\frac{dx}{dt} = A(t)x + \left(I + \frac{1}{m}A(t)\right)^{-1} f(t)$$

for any integer $m > 2H^{-1}$ has a unique generalized bounded solution, whence the uniqueness of the generalized solution of equation (1) follows.

Theorem 2. *Suppose there exists a strongly continuous derivative $(A^{-1}(t))'$, which for each t is a linear bounded operator. If the generalized solution $x^*(t)$ takes values in D and is a continuously differentiable function, then this solution is a classical bounded solution of equation (1).*

If $A(t) = A$ is a constant operator, then the requirement of complete continuity of the operator A^{-1} may be omitted. Suppose that the resolvent of the operator $-A$ satisfies the condition

$$\|(\lambda I + A)^{-1}\| \leq \frac{1}{\operatorname{Re}\lambda - \omega} \qquad (\operatorname{Re}\lambda > \omega), \tag{8}$$

where ω is some negative number. Then the existence of a bounded solution of equation (5) follows from the results of [2], with

$$x_n(t) = \int_t^{t+1} e^{-(t-s)A_n} F_n(s)\, ds,$$

where

$$F_n(t) = -f(t) - e^{-A_n} f(t+1) - e^{-2A_n} f(t+2) - \cdots.$$

Theorem 3. *Suppose that $f(t)$ takes values in the domain of definition of the operator A^2. Suppose that the function $A^2 f(t)$ is continuous on $(-\infty, \infty)$ and that*

$$\sup \|A^2 f(t)\| < +\infty.$$

If condition (8) holds for the operator A, then equation (1) has one and only one generalized bounded solution.

Proof. Since the operator A satisfies condition (4), using the identity of the form (7) for the functions $x_n(t)$ and the method of proof of Lemma 4, we establish that the functions $x_n(t)$ form a fundamental sequence.

The concept of a generalized solution bounded on $(-\infty, \infty)$ can also be introduced for a system of the form

$$dx/dt = A(t)x + f(t, x, y), \qquad dy/dt = B(t)y + g(t, x, y), \tag{9}$$

where $A(t)$ and $B(t)$ are, for each t, unbounded linear operators with dense domains of definition in the Banach spaces E_1 and E_2, respectively.

It is assumed that the operators $A(t)$, $B(t)$ and the functions $f(t, x(t), y(t))$, $g(-t, x(-t), y(-t))$, for any uniformly continuous and bounded functions $x(t)$ and $y(t)$ on $(-\infty, \infty)$, ensure the existence and uniqueness of generalized bounded solutions of the following equations:

$$\begin{aligned} du/dt &= A(t)u + f(t, x(t), y(t)), \\ dv/dt &= -B(-t)v - g(-t, x(-t), y(-t)). \end{aligned} \tag{10}$$

In the space of ordered pairs of the form $(x(t), y(t))$, where $x(t)$ and $y(t)$ are uniformly continuous and bounded on $(-\infty, \infty)$ functions with values respectively in E_1 and E_2, introduce an operator W, which assigns to the pair $(x(t), y(t))$ the pair $(u(t), v_1(t))$, where $u(t)$ and $v(t) = v_1(-t)$ are generalized bounded solutions, respectively, of equations (10).

Definition 2. A fixed point of the operator W is called a generalized bounded solution of system (9).

Using various conditions for the existence of a fixed point of the operator W, one can obtain conditions for the existence of a generalized bounded solution of system (9), as well as conditions under which this solution is a true bounded solution.

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Note: Figure translations are in progress. See original paper for figures.