DEFECT INDICES OF SYMMETRIC DIFFERENTIAL OPERATORS WITH POLYNOMIAL COEFFICIENTS

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

UDC 517.948+513.88

MATHEMATICS

A. L. CHISTYAKOV

DEFECT INDICES OF SYMMETRIC DIFFERENTIAL OPERATORS WITH POLYNOMIAL COEFFICIENTS

(Presented by Academician I. G. Petrovskii, 16 X 1968)

Let L_N be a minimal symmetric differential operator in the space of square-integrable functions $L_2(R^N)$, where R^N is N-dimensional Euclidean space. The coefficients of the differential expressions are assumed to depend polynomially on $x_1, x_2, \dots, x_N$.

In the present paper the defect indices of the operators L_N are investigated. In particular, for ordinary differential operators of order m (on the entire axis $-\infty < x < \infty$) conditions are obtained under which the defect indices take the values (ρ, ρ), where ρ is any integer from the interval $[0, m]$. For operators in partial derivatives (L_N, $N \geq 2$), sufficient conditions for self-adjointness are obtained.

In studying differential operators with polynomial coefficients it proves convenient to consider them in a basis of multidimensional Hermite functions. Then the operators are realized in the form of infinite Jacobi matrices with matrix coefficients. The defect indices of such operators were investigated by the author earlier ([1]).

§ 1. **Ordinary differential operators.** Let l be a self-adjoint differential expression of order m on the line $R^1 = (-\infty, \infty)$ of the form

$$l = \sum_{s+t \leq m} p_{st} \left(x - \frac{d}{dx}\right)^s \left(x + \frac{d}{dx}\right)^t, \tag{1}$$

where s and t are nonnegative integers, and the coefficients p_{st} satisfy the relations $p_{ts} = \overline{p}_{st}$. It is clear that every differential expression with polynomial coefficients can be put into the form (1). The minimal differential operator L (or L_1) is the closure of the operator L', defined by the expression $L'f = lf$ on finite, infinitely differentiable functions.

Theorem 1. *The minimal differential operator L, generated by the expression (1), is self-adjoint if $m = 2r$ and*

$$|p_{rr}| > \sum_{|\sigma|=1}^{r} |p_{r-\sigma,\,r+\sigma}|. \tag{2}$$

Proof. The operator L coincides with the operator C, the closure of the operator C' generated by the expression (1) on the linear span of the Hermite functions $\psi_n(x)$ $(n = 0, 1, \ldots)$. The proof of this fact is analogous to that given in ([2]), p. 299, for the Laplace operator. The corresponding matrix form of the operator C in the basis $\psi_0(x), \psi_1(x), \ldots$ has the form of a Hermitian matrix $C = \{c_{ki}\}_{k,i=0}^{\infty}$ possessing the finiteness property ($c_{k,k+j} = 0$ for $|j| > m = 2r$). The nonzero elements of this matrix have the asymptotic behavior, as $k \to \infty$,

$$c_{k,k+j} = \begin{cases} p_{r-\sigma,\,r+\sigma}(2k)^r[1 + O(k^{-1/2})], & \text{for } j = 2\sigma \ (|\sigma| = 0, 1, \ldots, 2r), \\ O(k^{r-1/2}), & \text{for odd } j,\ |j| < 2r. \end{cases} \tag{3}$$

This follows from the fact that the action of the operators $a = \frac{1}{\sqrt{2}}(x + d/dx)$ and $a^* = \frac{1}{\sqrt{2}}(x - d/dx)$ on the basis vectors is expressed by the formulas

$$a\psi_n(x) = \sqrt{n}\psi_{n-1}(x), \qquad a^*\psi_n(x) = \sqrt{n+1}\psi_{n+1}(x).$$

Sufficient conditions for self-adjointness of the operator C, formulated in ([1]), are expressed by the inequality

$$\mu(k_0, \lambda) = \sup_{k \geq k_0} |c_{kk} - \lambda|^{-1} \sum_{|j|=1}^{m} |c_{k,k+j}| < 1$$

for some sufficiently large k_0 and $\lambda = \pm i$. In our case, for large k we obtain

$$|c_{kk} - \lambda|^{-1} \sum_{|j|=1}^{m} |c_{k,k+j}| = |p_{rr}|^{-1}(2k)^{-r}\{1 + O(k^{-1/2})\} \left\{ \sum_{|\sigma|=1}^{r} |p_{r-\sigma,r+\sigma}|(2k)^r + O(k^{r-1/2}) \right\}.$$

Condition (2) makes it possible to choose k_0 so that $\mu(k_0, \pm i) < 1$. Theorem 1 is proved.*

More precise results are obtained by means of the asymptotic method developed by I. M. Rapoport ([6]) for systems of first-order difference equations having the quasidiagonal form $t_{k+1} = (\Gamma_k + \Gamma_{kD}k)t_k$, where Γ_k is a diagonal matrix, and the elements $d_{ij}(k)$ of the matrix D_k satisfy the condition

$$\sum_{k=0}^{\infty} |d_{ij}(k)| < \infty.$$

We have managed to supplement I. M. Rapoport' s results concerning the reduction of systems to quasidiagonal form, while computing the asymptotics of the transformation matrix as $k \to \infty$. This made it possible to compute the deficiency indices of the operator L from the asymptotics of the roots of the characteristic equation (as $k \to \infty$)

$$\sum_{\sigma=0}^{m} w^{2\sigma}\left[p_{m-\sigma,\sigma} + k^{-1}\left(N_\sigma p_{m-\sigma,\sigma} + \frac{1}{2}p_{m-\sigma-1,\sigma-1}\right)\right] + (2k)^{-1/2}\sum_{\mu=1}^{m} w^{2\mu-1}p_{m-\mu,\mu-1} = 0. \tag{4}$$

The limiting values of these roots as $k \to \infty$ are described by the equation

$$\sum_{\sigma=0}^{m} w^{2\sigma}p_{m-\sigma,\sigma} = 0. \tag{5}$$

Theorem 2. *The minimal differential operator L, generated by expression (1) for $m > 2$, has deficiency indices $(0, \rho)$, if:*

1) $p_{0,m} \neq 0$;
2) the roots of equation (5) are distinct;
3) $m - \rho$ roots of equation (4), for large k, have the estimate

$$|w(k)| \geq 1 - \frac{1}{2k} + O(k^{-\tau}), \qquad \tau > 1,$$

and the remaining $m + \rho$ roots have the estimate

$$|w(k)| \leq 1 - \nu k^{-1}$$

for some $\nu > 1/2$.

The proof of Theorem 2 reduces to Theorem 4 of ([1]). For this it is necessary to show that: 1) the quantities $f_s(k)$; $f_\sigma(k) = c_{k,k+m-\sigma}c_{k,k+m}^{-1}$, $\sigma \neq m$, and $f_m(k) = (c_{kk} - \bar{\lambda})c_{k,k+m}^{-1}$ satisfy the condition

$$\sum_{k=0}^{\infty} |f_s(k+1) - f_s(k)| < \infty$$

and 2) the roots of equation (4) coincide (up to

accuracy

* In the case $r = 1$, the operator L is self-adjoint for arbitrary p_{st}, which follows from conditions for self-adjointness of Carleman type (see ([3–5])) and estimates (3).

up to $O(k^{-2})$, with the roots of the equation

$$\sum_{j=-m}^{m} c_{k,k+j} w^{m+j} = \overline{\lambda} w^m,$$

which is equivalent to the equation

$$\sum_{s=0}^{2} f_s(k) w^{2m-s} = 0.$$

Both required conditions follow from a more exact formula for the matrix elements $c_{k,k+j}$ than that indicated in formula (3), and from condition 1).

An obvious consequence of this theorem is the following assertion.

Theorem 3. *The minimal differential operator* $\mathbf{L}$*, generated by expression (1) for* $m = 2r$ $(r > 1)$*, is self-adjoint if* $p_{0,m} \neq 0$*, the roots of equation (5) are distinct and* m *of them lie inside the unit circle, while the remaining* m *lie outside it.*

The most difficult case in the asymptotic method is when equation (5) has multiple roots on the unit circle. Such a problem is solved below for operators of a special form. In doing so, necessary and sufficient conditions are obtained for the self-adjointness of the operator $\mathbf{L}$.

Theorem 4. *The minimal differential operator* $\mathbf{L}$*, generated by the expression*

$$h_r \left(x - \frac{d}{dx}\right)^{2r} + h_0 \left(x - \frac{d}{dx}\right)^{r} \left(x + \frac{d}{dx}\right)^{r} + \overline{h}_r \left(x + \frac{d}{dx}\right)^{2r} \quad (r > 1),$$

is self-adjoint if and only if $|h_0| > 2|h_r|$*. For* $|h_0| \leq 2|h_r|$ *this operator has defect indices* $(2r, 2r)$*.*

§ 2. **Differential operators in partial derivatives.** Let l_N be a self-adjoint differential expression of order m (in the space R^N) of the form

$$l_N = \sum_{s+t \leq m} \sum_{n_1+\dots+n_N=s} \sum_{q_1+\dots+q_N=t} p_{st}(n_1, \dots, n_N \mid q_1 \dots q_N) \times$$

$$\times \left(x_1 - \frac{\partial}{\partial x_1}\right)^{n_1} \cdots \left(x_N - \frac{\partial}{\partial x_N}\right)^{n_N} \left(x_1 + \frac{\partial}{\partial x_1}\right)^{q_1} \cdots \left(x_N + \frac{\partial}{\partial x_N}\right)^{q_N}, \quad (6)$$

where the coefficients are connected by the relations

$$p_{st}(n_1, n_2, \dots, n_N \mid q_1, q_2, \dots, q_N) = \overline{p}_{ts}(q_1, q_2, \dots, q_N \mid n_1, n_2, \dots, n_N).$$

We introduce the minimal operators generated by the differential expressions**

$$a_k^* = \frac{1}{\sqrt{2}} \left(x_k - \frac{\partial}{\partial x_k}\right), \qquad a_k = \frac{1}{\sqrt{2}} \left(x_k + \frac{\partial}{\partial x_k}\right). \quad (7)$$

As is not difficult to see, every expression of the form (6) can be reduced to the form

$$l_N = \sum_{s+t\le m} \sum_{\xi_1=1}^{N} \cdots \sum_{\xi_s=1}^{N} \sum_{\eta_1=1}^{N} \cdots \sum_{\eta_t=1}^{N} v_{st}(\xi_1, \dots, \xi_s \mid \eta_1, \dots, \eta_t) a^*_{\xi_1} \cdots a^*_{\xi_s} a_{\eta_1} \cdots a_{\eta_t}, \tag{8}$$

where the functions v_{st} are symmetric separately in the parameters ξ_i and η_i. The relation between the coefficients p_{st} and v_{st} is given by the formula

$$p_{st}(n_1, n_2, \dots, n_N \mid q_1, q_2, \dots, q_N)\, 2^{\frac{1}{2}(n_1+\cdots+n_N+q_1+\cdots+q_N)} =$$

$$= \frac{(n_1+\cdots+n_N)!(q_1+\cdots+q_N)!}{n_1!n_2!\cdots n_N!\, q_1!q_2!\cdots q_N!}\, v_{st}(\underbrace{1,\dots,1}_{n_1},\dots,\underbrace{N,\dots,N}_{n_N} \mid \underbrace{1,\dots,1}_{q_1},\dots,\underbrace{N,\dots,N}_{q_N}).$$

---

* It is not difficult to show that Theorem 1 is a consequence of Theorem 3.

** In quantum mechanics of systems with a variable number of particles, these operators are called creation and annihilation operators. They satisfy the Bose commutation relations $[a_j, a^*_k] = a_j a^*_k - a^*_k a_j = \delta_{kj}$, $[a_j, a_k] = [a^*_j, a^*_k] = 0$, where δ_{kj} is the Kronecker symbol.

The following two assertions are proved on the basis of Theorem 1 from the paper (1).

Theorem 5. The minimal differential operator $\mathbf{L}_N$, generated by expression (8) (where a^*_k and a_k are defined by formulas (7)), is self-adjoint under the following set of conditions:

1) $m = 2r$;

2)

$$v_{rr}(\xi_1, \xi_2, \dots, \xi_r \mid \eta_1, \eta_2, \dots, \eta_r) = v_r(\xi_1, \dots, \xi_r)\delta_{\xi_1\eta_1}\delta_{\xi_2\eta_2}\cdots\delta_{\xi_r\eta_r};$$

3)

$$2\sum_{\sigma=1}^{r} K_{r-\sigma,r+\sigma} < \min_{\xi_1,\dots,\xi_r} v_r(\xi_1, \dots, \xi_r),$$

where K_{st} is the norm of the operator

$$\mathcal{K}_{st}\varphi_{\eta_1\dots\eta_t} = \sum_{\eta_1=1}^{N} \cdots \sum_{\eta_t=1}^{N} v_{st}(\xi_1, \dots, \xi_s \mid \eta_1, \dots, \eta_t)\varphi_{\eta_1\dots\eta_t}.$$

For example, the operator $\mathbf{L}_N$ generated by the expression

$$\sum_{k=1}^{N}\sum_{l=-r}^{r} d_k^{(l)} \left(x_k - \frac{\partial}{\partial x_k}\right)^{r+l} \left(x_k + \frac{\partial}{\partial x_k}\right)^{r-l} + P_{2r-1}\left(x_1, \dots, x_N, i\frac{\partial}{\partial x_1}, \dots, i\frac{\partial}{\partial x_N}\right),$$

is self-adjoint provided that $d_k^{(l)} = \bar{d}_k^{(-l)}$, P_{2r-1} is a polynomial of degree $2r-1$ with real coefficients, and

$$\min_k d_k^0 > 2\sum_{l=1}^{r} \max_k |d_k^{(l)}|.$$

Theorem 6. The minimal differential operator $\mathbf{L}_2$, generated by the expression

$$-\frac{\partial^2}{\partial x^2} + x^2 + x\left[id_1\frac{\partial}{\partial y} + \varphi_1(y)\right] - \gamma\frac{\partial^2}{\partial y^2} + id_2\frac{\partial}{\partial y} + \varphi_2(y),$$

is self-adjoint if $\gamma > 0$, $d_{1,2}$ are real numbers, and the functions $\varphi_{1,2}(y)$ belong to $L_2(R^1)$.

I express my sincere gratitude to F. A. Berezin, who drew my attention to the indicated circle of questions.

Institute of Organoelement Compounds
Academy of Sciences of the USSR

Received
16 X 1968

REFERENCES

1. A. L. Chistyakov, DAN, **178**, No. 5 (1968).
2. T. Kato, *Perturbation Theory for Linear Operators*, Berlin–Heidelberg–N. Y., 1966.
3. N. I. Akhiezer, *The Classical Moment Problem*, Moscow, 1961.
4. Yu. M. Berezanskii, *Expansion in Eigenfunctions of Self-Adjoint Operators*, Kiev, 1965.
5. F. A. Berezin, *The Method of Second Quantization*, "Nauka," 1965.
6. I. M. Rapoport, *On Certain Asymptotic Methods in the Theory of Differential Equations*, Kiev, 1954.