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Abstract

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Nonlinear Excitation of Drift Waves in an Inhomogeneous Plasma

As is known, in an inhomogeneous plasma in a strong magnetic field there exists a broad class of drift instabilities (see, for example, ([1], [2])). These instabilities are associated with the excitation of drift waves of small amplitude, and for their theoretical analysis the linear approximation is sufficient. At the same time, a quite natural question may arise as to whether drift waves of finite amplitude have any new qualitative features that are not covered by the linear approximation. It is clear that if the amplitude is not small, the wave becomes substantially nonlinear and requires a special nonlinear treatment. However, even at small amplitude there may arise effects connected with the fact that such a wave contains a new group of particles trapped in the potential wells of the wave. Such particles execute finite motion, i.e., in their properties they differ substantially from particles moving freely along the magnetic field (which we assume to be homogeneous) in the absence of the wave. The number of trapped particles is proportional to the square root of the wave amplitude, and therefore even in a wave of comparatively small amplitude their fraction may be quite appreciable.

It is known that magnetically trapped particles (on inhomogeneities of the magnetic field) can lead to a new class of instabilities ([3]). One may think that analogous effects should also appear for particles electrically trapped in a wave. Some such effects in a collisionless plasma were discussed in ([4]). In the present paper it will be shown that, under conditions where the electron collision frequency is of the order of the drift frequency and there is a temperature gradient in the plasma directed oppositely to the density gradient, excitation of a drift wave of finite amplitude by electrically trapped electrons becomes possible. The corresponding instability is similar to the instability due to magnetically trapped electrons in toroidal systems ([5]). It is associated with dissipation during the transition of electrons from trapped to passing states as a result of collisions.

Let us consider a plane layer of plasma with $\beta = 8\pi nT/B^2 \ll 1$, infinite in extent in the directions of the y and z axes and inhomogeneous in the x direction, so that the equilibrium densities n and temperature T are functions of x only. We shall regard the magnetic field $\mathbf{B}$ as homogeneous and directed along the z axis. Suppose that a drift wave of finite amplitude φ_0 propagates along such a layer. We shall assume that this wave is standing along the z axis, since for the mechanism of excitation of oscillations considered here it is important that the electrons be periodically trapped by the wave and then released (in a wave traveling along z such an effect is absent—the trapped particles are simply carried along the z axis together with the wave).

In accordance with this, we choose the potential of the electric-field perturbation of the wave in the form

$$\varphi = \varphi_0 e^{-i\omega t + ik_y y} \cos k_z z + \text{c.c.} \tag{1}$$

where ω is the oscillation frequency; k_y, k_z are the components of the wave vector along the y and z axes. We shall not consider the dependence of φ_0 on x, as is usual

is done in the approximation of local perturbations. In expression (1) the frequency ω should be regarded as real; however, in order to shorten the intermediate calculations it is convenient to include in ω an imaginary part $i\gamma$, corresponding to the growth increment of small perturbations, i.e., to put $\varphi_0 = \varphi_0(t=0)e^{\gamma t}$.

We shall assume that the amplitude of the oscillations is small, i.e. $e\varphi/T \ll 1$, so that in all expressions not referring to trapped particles one may restrict oneself to the linear approximation. Let us also assume that k_z is sufficiently small, so that the longitudinal motion of the ions may be neglected. In this approximation the perturbation of the ion density n_i' is found from the continuity equation, in which the transverse velocity is determined by the electric drift $\mathbf{v}_\perp = c[\mathbf{B}\nabla\varphi]/B^2$:

$$-in_i' - i\,\frac{ck_y\varphi}{B}\frac{dn}{dx} = 0. \tag{2}$$

Hence we find

$$\frac{n_i'}{n} = \frac{\omega^*}{\omega}\frac{e\varphi}{T}, \tag{3}$$

where ω^* is the so-called drift frequency:

$$\omega^* = -\frac{k_y cT}{eBn}\frac{dn}{dx}.$$

To find the perturbations of the electron density n_e' we shall use the drift approximation. In this approximation the kinetic equation for the electron distribution function f over the longitudinal velocity v_z has the form:

$$\frac{\partial f}{\partial t} + v_z \frac{\partial f}{\partial z} + \frac{e}{m} \frac{\partial \varphi}{\partial z} \frac{\partial f}{\partial v_z} + c \frac{[\mathbf{B} \nabla \varphi]}{B^2} \nabla_\perp f = St, \tag{4}$$

where St is the collision term, which we shall regard as being of order $\partial f / \partial t$. Let us assume, as usual, that the phase velocity of the wave along the magnetic field, ω/k_z, is considerably smaller than the thermal velocity of the electrons, $v_e = \sqrt{2T/m}$. Then in equation (4) the second and third terms are larger than the others, so that in the first approximation only these need be retained.

In the approximation linear in φ we then obtain $f' = (e\varphi/T) f_0$, where f_0 is the unperturbed Maxwellian distribution function. This part of the perturbation simply corresponds to the Boltzmann distribution; the collision term St also vanishes for it.

In the next approximation in $\omega/k_z v_e$ it is necessary to take into account the small first and fourth terms on the left-hand side of (4). In the linear approximation, for $\eta = d \ln T / d \ln n < 0$, we would obtain a small growth increment of the perturbations

$$\gamma \sim \frac{(\omega^*)^2}{k_z^2 v_e} \eta.$$

We shall show that, when trapped electrons are taken into account, a much faster excitation of drift waves is possible.

Let us represent the perturbation of the electron distribution function in the form

$$f' = \frac{e\varphi}{T} f_0 + f_t',$$

where the second term takes into account the non-Boltzmann part of the perturbation. To find f_t' one should take the second approximation of equation (4). It is not difficult to see that passing particles give a small contribution to f_t', of order $\omega/k_z v_e$, since it is acquired by averaging over z over many wavelengths. Trapped particles, on the other hand, are at all times in one and the same phase of the wave, and therefore they accumulate the perturbation. Consequently, neglecting small quantities of order $\omega/k_z v_e$, by f' one may understand the non-Boltzmann part of the distribution function only of the trapped particles. For this part the collision term may be approximated simply by the expression $-\nu_{\text{eff}} f_t'$, since the number of particles in the group under consideration is not conserved—they may pass into the Boltzmann-distributed part of the distribution function. If it is taken into account that the Coulomb collision term has a diffusive character (it contains the second derivative-

depending on the distribution function), while the trapping region has the order of magnitude $|v_z| \lesssim \sqrt{e\varphi_0/m}$, the effective frequency ν_{ef} should be considered approximately equal to $\nu_e T/e\varphi_0$, where ν_e is the average frequency of electron collisions for all electrons.

Taking into account everything said above and retaining only the linear part of the last term on the left-hand side of (4), we obtain for f_t' the equation

$$v_z \frac{\partial f_t'}{\partial z} + \frac{e}{m}\frac{\partial \varphi}{\partial z}\frac{\partial f_t'}{\partial v_z} = i(\omega - \hat{\omega}^*) f_0 \frac{e\varphi}{T} + i(\omega + i\nu_{ef}) f_t', \tag{5}$$

where $\hat{\omega}^* = \omega^*(1 - 1/2\eta)$ for $v_z \ll v_e$.

In equation (5) the right-hand side is small; therefore it is convenient to solve it by the method of successive approximations. To this end we pass to a new variable—the longitudinal velocity of the particle as a function of energy:

$$v_z = \left[\frac{2}{m}(E + e\varphi)\right]^{1/2}.$$

The operator on the left-hand side of (5) then becomes simply $v_z\partial/\partial z$, and the equation for f_t' takes the form

$$v_z \frac{\partial f_t'}{\partial z} = i(\omega - \hat{\omega}^*) f_0 \frac{e\varphi}{T} + i(\omega + i\nu_{ef}) f_t'. \tag{6}$$

In the first approximation $v_z\partial f_t'/\partial z = 0$, i.e., f_t' does not depend on z. From the solvability condition for the next approximation, which reduces to setting equal to zero the integral over dz, with weight $1/v_z$, of the right-hand side of (6), we obtain:

$$f_t' = -\frac{\omega - \hat{\omega}^*}{\omega + i\nu_{ef}}\, f_0 \frac{e\bar{\varphi}}{T}, \tag{7}$$

where

$$\bar{\varphi} = \frac{1}{\tau}\oint \frac{dz}{v_z}\varphi, \qquad \tau = \oint \frac{dv_z}{v_z}$$

is the period of oscillations of trapped particles between the crests of the wave. Integrating the distribution function (6) over velocities and taking into account that $dv_z = 2\,dE/m|v_z|$, we find an expression for the density of trapped electrons nn_t'.

The total perturbed electron density is equal to

$$n_e' = \frac{e\varphi}{T}n + n_t'.$$

From the quasineutrality condition, equating the perturbed densities of electrons and ions (3), we obtain an equation for the potential

$$\left(1-\frac{\omega^*}{\omega}\right)\varphi(z)-\frac{\omega-\hat{\omega}^*}{\omega+i\nu_{ef}}\frac{1}{\sqrt{\pi}v_e}\int\bar{\varphi}(E)\frac{2\,dE}{m|v_z|}=0, \tag{8}$$

where integration over dE is performed, for a given z, between the points where v_z is a real quantity. The interval of integration over E is of order $e\varphi_0/T \ll 1$. Therefore in equation (8) we have approximately substituted

$$f_0=\frac{1}{\sqrt{\pi}v_e}e^{-E/T}\simeq\frac{1}{\sqrt{\pi}v_e}$$

and taken f_0 outside the integral.

Equation (8) is a nonlinear integral equation for the potential. Its solution could be sought by expanding φ in harmonics. However, if one is not interested in the spatial structure of φ, then the solvability condition for (8), which replaces the dispersion equation of linear theory, may be obtained approximately as follows. Multiply (8) by φ^* and integrate over z over one period. As a result we obtain the integral relation:

$$\left(1-\frac{\omega^*}{\omega}\right)\int|\varphi(z)|^2\,dz-\frac{\omega-\hat{\omega}^*}{\omega+i\nu_{ef}}\frac{1}{\sqrt{\pi}}\int_{e\varphi_{\min}}^{e\varphi_{\max}}|\bar{\varphi}(E)|^2\frac{|\tau|\,dE}{mv_e}=0. \tag{9}$$

Here in the second term we have changed the order of integration over z and E. Into this relation one may substitute an approximate expression for φ, found—

for example, by choosing it in the form (1). After this, averaging (9) over y, it is not difficult to carry out the integrations indicated in (9). As a result we obtain the following "dispersion" equation:

$$1-\frac{\omega^*}{\omega}-\alpha\left(\frac{e\varphi_0}{T}\right)^{1/2}\frac{\omega-\hat{\omega}^*}{\omega+i\nu_{ef}}=0; \tag{10}$$

here the factor $(e\varphi_0/T)^{1/2}$ takes into account the fraction of electrons trapped by the wave;

$$\alpha=\left(\frac{2}{\pi}\right)^{3/2}2\int_0^1\frac{(2E(\varkappa)-K(\varkappa))^2}{K(\varkappa)}\,d\varkappa^2$$

—is a numerical constant ~ 3;

$E(\varkappa)$ and $K(\varkappa)$ are complete elliptic integrals of the first and second kind,

$$\omega^* = -\frac{k_y cT}{eB}\frac{d\ln n}{dx}, \qquad \hat{\omega}^* = \omega^*(1 - {}^1\!/_2\eta), \qquad \eta = \frac{d\ln T}{d\ln n}, \qquad \nu_{ef} = \nu_e\frac{T}{e\varphi_0}.$$

Equation (10) is very similar to the dispersion equation for an instability on magnetically trapped electrons (5). Taking into account that the right-hand side of (10) is small for $e\varphi_0/T \ll 1$, it is easy to obtain an expression for the frequency and growth rate of the oscillations

$$\omega \simeq \omega^*, \qquad \gamma = -\frac{\alpha}{2}\eta\left(\frac{e\varphi_0}{T}\right)^{1/2}\frac{(\omega^*)^2}{(\omega^*)^2 + (\nu_{ef})^2}\,\nu_{ef} \tag{11}$$

for $\eta < 0$, $\gamma > 0$, i.e., growing solutions are possible. Here γ should be understood as an abbreviated notation for $\frac{1}{\varphi_0}\frac{\partial\varphi_0}{\partial t}$. Taking this circumstance into account and using (11), we obtain an equation describing the behavior of the amplitude with time:

$$\frac{\partial y}{\partial t} = \frac{\alpha}{2}|\eta|\,\frac{y^{3/2}}{y^2 + (\nu_e/\omega^*)^2}\,\nu_e \qquad \left(y = \frac{e\varphi_0}{T} \ll 1\right). \tag{12}$$

Investigation of this equation shows that the characteristic time of growth of the wave amplitude t_* (the reciprocal of which may be regarded as the "growth rate" of the nonlinear instability under consideration) always satisfies the condition $\omega^* t_* \gg 1$ (if the initial amplitude is small, $y(0) \ll 1$). In this sense the instability under consideration has a small growth rate. The maximum value γ_{nonlin} is attained for $\nu_e \sim \omega^* y(0)$; in this case $\gamma_{\text{nonlin}} \sim |y(0)|\omega^*$. However, the growth rate of the instability under consideration may substantially exceed the growth rate of the linear theory. From the condition $\gamma_{\text{nonlin}} \gg \gamma_l \sim (\omega^*)^2/k_z\nu_e$ we find that this takes place for

$$e\varphi_0(0)/T \gg (\omega^*/k_z\nu_e)^2 \geq m_e/m_i,$$

i.e., even at a sufficiently small initial noise level.

Thus, we have shown that under certain conditions a weakly nonlinear drift wave may grow in time much faster than in the linear approximation. Although the conditions considered here ($d\ln T/d\ln n < 0,\ \nu_e \sim \omega^* y(0)$) are rather specific, the very fact of the existence of nonlinear pumping is very interesting and suggestive. It shows that nonlinear waves with trapped particles may possess new physical properties requiring, in each particular case, a more detailed investigation.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.