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# EMBEDDING COMPACTA IN EUCLIDEAN SPACE

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**Abstract**

**Full Text**

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**MATHEMATICS**

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## EMBEDDING COMPACTA IN EUCLIDEAN SPACE

*(Presented by Academician P. S. Aleksandrov on 13 XII 1968)*

1. An embedding of a space  $X$  in a space  $Y$  is an arbitrary homeomorphic mapping  $X \rightarrow Y$ ; we shall omit the notation of the mapping and by an embedding understand simply the inclusion  $X \subset Y$ .

Here we shall obtain some results for embeddings of compacta of dimension  $\leq n - 3$  in Euclidean space  $E^n$ . The basic concept is the embedding dimension of a compactum  $K$  in the space  $E^n$ ,  $K \subset E^n$ , denoted by  $\text{dem } K$ , ( $\text{dem}$  = dimension of embedding), which is an integer-valued invariant, unchanged under homeomorphisms of  $E^n$  onto itself (nonnegative for nonempty compacta), reminiscent of the characterization of the ordinary topological dimension  $\dim K$  of a compactum  $K$  by P. S. Aleksandrov's method of approximating compacta by polyhedra (see <sup>(1)</sup>). In the case  $\dim K \leq n - 3$  this invariant depends on the embedding of  $K$  in  $E^n$ .

We shall call an embedding  $K \subset E^n$ , where  $\dim K \leq n - 3$ , tame if  $\text{dem } K = \dim K$ , and wild otherwise. (We note that always  $\text{dem } K \geq \dim K$ .)

In the case of zero-dimensional compacta and  $r$ -dimensional topological polyhedra in  $E^n$  ( $2r + 2 \leq n$ ), the definition of a tame embedding introduced by us coincides with the previously known one (see <sup>(2)</sup>). For nontrivial dimensions it is unknown whether the concept of a tame embedding introduced by us coincides with the classical one (see unsolved questions III, III'). Menger's problem on universal compacta is also considered (see <sup>(3)</sup>), which turns out to be connected with embeddings.

Let us recall some simplest notation and concepts.  $U(K, \varepsilon)$  denotes the  $\varepsilon$ -neighborhood of the compactum  $K$  in  $E^n$ . A compactum  $K$  lying in  $E^n$  is called cellularly separated if in every neighborhood  $U$  of the compactum  $K$  one can inscribe such a neighborhood  $V$  that  $U \supset \bar{V} \supset V \supset K$ , and  $\bar{V}$  is homeomorphic to a disjoint union of a finite number of topological cubes. A homotopy  $F_t : X \rightarrow X$ ,  $t \in [0, 1]$ , is called a pseudoisotopy if for every  $t < 1$ ,  $F_t$  is a homeomorphism of the space  $X$  onto itself, and an isotopy if  $F_t$  is a homeomorphism for every  $t$ .

**Definition 1.** The **embedding dimension of a nonempty compactum**  $K \subset E^n$  is the least integer  $\text{dem } K = r \geq 0$  such that for every  $\varepsilon > 0$  there exists an  $\varepsilon$ -pseudoisotopy  $F_t : E^n \rightarrow E^n$ ,  $t \in [0, 1]$ , of the space  $E^n$  onto itself, fixed outside  $U(K, \varepsilon)$ , and such that  $F_1(K) = P$ , where  $P$  is a rectilinear polyhedron in  $E^n$ ,  $\dim P = r$ .

The embedding dimension of the empty compactum in  $E^n$  is by definition equal to  $-1$ .

**Main theorem.** If  $K \subset E^n$ ,  $n \neq 3$ ,  $\dim K \geq n - 2$ , then always  $\text{dem } K = \dim K$ ; if  $n \geq 5$  and  $\dim K \leq n - 3$ , then the embedding dimension assumes two (and only two) values: either  $\dim K$ , or  $n - 2$ , the latter occurring if and only if  $E^n \setminus K$  does not have the property 1-*ULC*.

**2. Definition 2.** The **dual embedding dimension of a nonempty compactum**  $K \subset E^n$  is called the least

an integer  $\text{Dem } K = r \geq 0$  such that, whatever rectilinear polyhedron  $P \subset E^n$ ,  $\dim P \leq n - r - 1$ , for every  $\varepsilon > 0$  there exists an  $\varepsilon$ -isotopy  $F_t : E^n \rightarrow E^n$ ,  $t \in [0, 1]$ , fixed outside  $U(K \cap P, \varepsilon)$  and such that  $F_1(K) \cap P = \Lambda$ . The dual embedding dimension of the empty compactum in  $E^n$  is, by definition, equal to  $-1$ .

Let  $M^n$  be the geometric cube in  $E^n$ . Divide the cube  $M^n$  into  $3^n$  equal cubes of first rank by hyperplanes drawn perpendicular to the edges of the cube  $M^n$  at the points dividing the edges into three equal parts; from these  $3^n$  cubes select those which intersect the  $r$ -dimensional skeleton of the cube  $M^n$ , and denote their union by  $M_{r,1}^n$ . Each cube entering into  $M_{r,1}^n$  is divided in an analogous manner into  $3^n$  cubes of second rank; the sum of all the analogously selected cubes of second rank is denoted by  $M_{r,2}^n$ , and so on. We obtain a decreasing sequence of closed sets

$$M^n \supset M_{r,1}^n \supset M_{r,2}^n \supset \dots$$

The compactum  $M_r^n = \bigcap_{k=1}^{\infty} M_{r,k}^n$  is the Menger compactum (see (3)).

For a compactum  $K \subset E^n$  the following assertions follow from the definitions or were known earlier.

- (a) Each of the conditions  $\text{dem } K = 0$  or  $\text{Dem } K = 0$ , respectively, is equivalent to the conjunction of two conditions: (1<sup>0</sup>)  $\dim K = 0$ ; (2<sup>0</sup>)  $K$  is cellularly decomposable.
- b) If  $n \geq 3$  and  $A_0$  is a wild zero-dimensional compactum in  $E^n$ ,  $A_0 \subset E^n$ , then  $\text{dem } A_0 = \text{Dem } A_0 = n - 2$ .
- (c) There exists a compactum  $K$  in  $E^3$  such that  $\dim K = 1$ , whereas  $\text{dem } K = \text{Dem } K = 2$ , for example, the compactum constructed by Bothe (4).

(d) The embedding dimensions  $\text{dem } K$ ,  $\text{Dem } K$  possess the properties of invariance and monotonicity, i.e. the corresponding value of the embedding dimension does not change if the space is subjected to a homeomorphism onto itself, and, on passing to a subcompactum, the corresponding value of the embedding dimension does not increase.

(e)  $\text{dem}(M_r^n) = \text{Dem}(M_r^n) = \dim(M_r^n) = r$ ,  $M_r^n \subset E^n$ .

3. The **main theorem** can be formulated as follows:

If a compactum  $K$  lies in  $E^n$ ,  $n \neq 3$ ,  $\dim K \geq n - 2$ , then always  $\text{dem } K = \dim K$ ; if  $n \geq 5$  and  $\dim K \leq n - 3$ , then the compactum  $K$  is tame in  $E^n$  if and only if  $E^n \setminus K$  has the property 1-*ULC*; but if  $E^n \setminus K$  does not have the property 1-*ULC*, then always  $\text{dem } K = n - 2$ .

**Remark.** By virtue of Bing's theorem this assertion is also true for  $n = 3$  and  $\dim K = 0$  (5). An analogous assertion was earlier proved by McMillan for zero-dimensional compacta in  $E^n$ ,  $n \geq 5$  (6).

**Theorem 1.** For every compactum  $K$  lying in  $E^n$ ,  $\text{dem } K = \text{Dem } K$ .

**Theorem 2.** Let compacta  $K$  and  $C$  lie in the space  $E^n$ ,  $n \geq 5$ ,  $\dim K = r$ ,  $2r + 2 \leq n$ , and let  $h : K \rightarrow C$  be a homeomorphism; if  $E^n \setminus K$  and  $E^n \setminus C$  have the property 1-*ULC*, then for every  $\varepsilon > \rho(h, 1)$  there exists an  $\varepsilon$ -isotopy  $F_t : E^n \rightarrow E^n$ ,  $t \in [0, 1]$ , fixed outside the prescribed neighborhood  $V \supset X(K, h, C)$  and such that  $F_1|_K = h$ . (Here  $X(K, h, C)$  denotes the union of the segments  $xh(x)$ ,  $x \in K$ .)

Let us note a particular case of Theorem 2, obtained earlier by Bryant and Seebeck: let a topological polyhedron  $P$  lie in  $E^n$ ,  $n \geq 5$ ,  $\dim P = r$ ,  $2r + 2 \leq n$ ; if  $E^n \setminus P$  has the property 1-*ULC*, then for every  $\varepsilon > 0$  there exists an  $\varepsilon$ -isotopy of the space  $E^n$  onto itself, fixed outside the  $\varepsilon$ -neighborhood of the polyhedron  $P$  and carrying it into a piecewise-linear one (7).

**Theorem 3.** If a compactum  $K$  lies in  $E^n$ , then the condition  $\text{Dem } K = r$  is equivalent to the fact that  $M_r^n \subset E^n$  is a Menger compactum of the least dimension for which there exists an isotopy  $F_t : E^n \rightarrow E^n$ ,  $t \in [0, 1]$ , fixed outside some bounded domain  $V$  and such that  $F_1(K) \subset M_r^n$ .

A somewhat weaker assertion was proved earlier by Bothe (8).

**Corollary 1.** The Menger compactum  $M_r^n$ ,  $M_r^n \subset E^n$ ,  $n \geq 5$ ,  $r \leq n - 3$ , is universal for all  $r$ -dimensional compacta in  $E^n$ ,  $K \subset E^n$ , such that  $E^n \setminus K$  has the property 1-*ULC*; moreover, in this case there exists an isotopy  $F_t : E^n \rightarrow E^n$ ,  $t \in [0, 1]$ , fixed outside some bounded domain  $V$  and such that  $F_1(K) \subset M_r^n$ .

**Corollary 2.** The Menger compactum  $M_r^n$ ,  $M_r^n \subset E^n$ ,  $n \geq 5$ ,  $r \leq n - 3$ , is universal for all  $r$ -dimensional compacta  $K$  lying in the hyperplane  $E^{n-1}$ ,  $K \subset E^{n-1} \subset E^n$ ; moreover, in this case there exists an isotopy  $F_t : E^n \rightarrow E^n$ ,  $t \in [0, 1]$ , fixed outside some bounded domain  $V$  and such that  $F_1(K) \subset M_r^n$ .

4. For the proof of the main theorem the absorption lemma <sup>(9)</sup> is used. We briefly describe the idea of the proof.

We shall prove that if  $\dim K \leq n-3$  and  $E^n \setminus K$  has the property 1-*ULC*, then  $\text{Dem } K \leq \dim K^*$ ; hence, taking into account that always  $\text{Dem } K \geq \dim K$ , we shall have  $\text{Dem } K = \dim K$ . All the remaining assertions of the theorem follow easily from the definitions. The proof of the indicated assertion is carried out by induction on the increasing dimension of the polyhedron  $P$ ,  $\dim P = p$ , from which, according to Definition 2, the compactum  $K$  is removed,  $0 \leq p \leq n-r-1$ , where  $r = \dim K$ ,  $n \geq 5$ . Since  $\dim K \leq n-3$ , the inductive assertion is fulfilled for the values  $p = 0, 1$ . Assuming that it is fulfilled for some  $p$ ,  $0 \leq p < n-r-1$ , we prove that it is fulfilled for  $p+1$ .

Let  $P$  be a rectilinear polyhedron in  $E^n$ ,  $\dim P = p+1$ . By the induction hypothesis we may assume that  $K \cap P^p = \Lambda$ , where  $P^p$  is the  $p$ -skeleton of the polyhedron  $P$  in some  $\varepsilon$ -triangulation of the polyhedron  $P$ . Let  $\Delta^{p+1}$  be some simplex in the given  $\varepsilon$ -triangulation of the polyhedron  $P$  such that  $\Delta^{p+1} \cap K \neq \Lambda$ ; since  $\partial(\Delta^{p+1}) \cap K = \Lambda$ , there exists a simplex  $\Delta_0^{p+1}$ , concentric with the simplex  $\Delta^{p+1}$ ,  $\text{int } \Delta^{p+1} \supset \Delta_0^{p+1}$ , and, if  $C = \Delta^{p+1} \setminus \text{int } \Delta_0^{p+1}$ , then  $C \cap K = \Lambda$ .

Let  $L^n(\Delta^{p+1}) = L^n$  be a convex polyhedral  $n$ -cube in  $E^n$  such that  $L^n \cap P = \Delta^{p+1}$ ,  $\partial \Delta^{p+1} \subset \partial L^n$ ,  $\text{int } \Delta^{p+1} \subset \text{int } L^n$ . Since  $E^n \setminus K$  has the property 1-*ULC* and  $p \leq n-r-2$ , one can show that the sphere  $\partial \Delta_0^{p+1}$  contracts to a point in the set  $(\text{int } L^n) \setminus K$ ; therefore there exists a polyhedral homotopy film  $\Pi_0$  in  $(\text{int } L^n) \setminus K$ , effecting the contractibility of the  $p$ -dimensional sphere  $\partial \Delta_0^{p+1}$  in  $(\text{int } L^n) \setminus K$  to a point; putting it in general position, we may assume that the self-intersection of this film has minimal dimension. If the self-intersection is empty and  $\Pi_0 \cap C = \partial \Pi_0 = \partial \Delta_0^{p+1}$ , then the resulting piecewise-linear disk  $\Pi = \Pi_0 \cup C$  forms a piecewise-linear pair of cells  $(L^n, \Pi)$ ; if it is unknotted, then  $\Pi$  is carried onto  $\Delta^{p+1}$  fixed outside  $\text{int } L^n$ , whereby the compactum  $K$  is removed from  $\Delta^{p+1}$ . If, however, the self-intersection of  $\Pi_0$  is nonempty, then either the absorption lemma is applied in a special way, or Ziman's device (with cones) is applied directly to eliminate the self-intersection.

**5. Questions.** (I). Let a compactum  $K$  lie in  $E^n$  and  $\dim K \leq n-3$ . Does there exist a homeomorphism  $h : K \rightarrow E^n$  such that  $E^n \setminus h(K)$  has the property 1-*ULC*?

(I'). Let a compactum  $K$  lie in  $E^n$  and  $\dim K \leq n-3$ . Does there exist, for every  $\varepsilon > 0$ , a homeomorphic  $\varepsilon$ -shift  $h_\varepsilon : K \rightarrow E^n$  such that  $E^n \setminus h(K)$  has the property 1-*ULC*?

(II). Let  $P$  be a topological polyhedron in  $E^n$ ,  $\dim P \leq n-3$ . Does there exist a piecewise-linear embedding  $h : P \rightarrow E^n$ ?

(II'). Let  $P$  be a topological polyhedron in  $E^n$ ,  $\dim P \leq n-3$ . Does there exist, for every  $\varepsilon > 0$ , a homeomorphic piecewise-linear  $\varepsilon$ -shift  $h_\varepsilon : P \rightarrow E^n$ ?

\* According to Theorem 1, in the formulation of the main theorem we may

replace  $\text{dem } K$  by  $\text{Dem } K$ .

(III). Let  $P$  be a topological polyhedron in  $E^n$ ,  $\dim P \leq n-3$ , and suppose that  $E^n \setminus P$  has the property 1-*ULC*. Will the polyhedron  $P$  be tame in  $E^n$ ?

(III'). Let  $P$  be a topological polyhedron in  $E^n$ ,  $\dim P \leq n-3$ , and suppose that  $E^n \setminus P$  has the property 1-*ULC*. Can the polyhedron  $P$  be carried into a piecewise-linear one by means of an  $\varepsilon$ -isotopy of the space  $E^n$ , fixed outside the  $\varepsilon$ -neighborhood of  $P$ ?

(We note that questions I–III' have positive solutions in trivial dimensions.)

(IV). Let a compactum  $K$  lie in  $E^n$ ,  $K \subset E^n$ ,  $n = 4$ . Will the equality  $\dim K = \text{Dem } K$  hold when  $\dim K = 0$  or 1 and  $E^n \setminus K$  has the property 1-*ULC*?

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