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L. G. MIKHAILOV, B. M. BILMAN, A. V. ZAMOTA

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Abstract

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MATHEMATICS

L. G. MIKHAILOV, B. M. BILMAN, A. V. ZAMOTA

MULTIDIMENSIONAL INTEGRAL EQUATIONS WITH DIFFERENCE KERNELS IN A HALF-STRIP

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The theory of one-dimensional integral equations with difference kernels on a semi-infinite interval, also called Wiener–Hopf equations (see, for example, ([1])), has been developed rather fully. For analogous multidimensional equations there are only isolated results. Thus, in ([2], [3]) an equation in a half-space is studied. It is shown that, under Noetherian conditions, such an equation is unconditionally uniquely solvable in a series of Banach spaces. In ([4], [5]) multidimensional integral equations with difference kernels in cones are considered. In the first of these works, conditions are given under which the equation has a unique solution, while in the second necessary and sufficient Noetherian conditions are indicated, the index being equal to zero. In all the works listed it is required that the kernel be summable over the whole space.

In the present work we shall consider equations with summability conditions in a strip. Such equations may have an index, unlike the equations considered in ([2–5]). Below, formulas are given for computing the number of solutions of the homogeneous equation and the number of solvability conditions for the nonhomogeneous one. To avoid cumbersome exposition we restrict ourselves to the three-dimensional case, which gives a complete idea of how the investigation is carried out in the case of an arbitrary number of dimensions.

Let Π denote the strip $0 \leq u_1 \leq 2\pi,\ 0 \leq u_2 \leq 2\pi,\ -\infty < u_3 < \infty$, and let Π_+ be the half-strip $u_3 \geq 0$. Let the kernel $k(u_1, u_2, u_3) \equiv k(u)$ be given in the strip Π and be summable,

$$Q \equiv \int_{\Pi} |k(u)|\, du < \infty. \tag{S}$$

Extend it periodically to the whole space, setting

$$k(u_1 + 2m\pi, u_2 + 2n\pi, u_3) = k(u_1, u_2, u_3), \qquad m, n = 0, \pm 1, \ldots$$

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The equation under consideration is*

$$f(x) = \int_{\Pi_+} k(x-y)f(y)\,dy + g(x), \qquad x \in \Pi_+, \tag{1}$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$, $dy = dy_1 dy_2 dy_3$. We shall study this equation in the following Banach spaces of complex-valued functions defined in Π_+: L_p is the space of measurable functions summable to the power p, $p \geq 1$; M is the space of measurable almost everywhere bounded functions; C is the subspace of functions $f(x_1, x_2, x_3)$ from M, continuous at finite points and having a limit $f(x_1, x_2, +\infty)$ as $u_3 \to +\infty$; C_0 is the subspace of functions $f(x_1, x_2, x_3)$ from C for which

* An equation on an arbitrary rectangular half-strip parallel to one of the coordinate axes is reduced to (1) by obvious changes of variables.

$$\lim_{x_3 \to +\infty} f(x_1, x_2, x_3) = 0.$$

For all these spaces we introduce the common notation E. The periodicity and summability conditions on the kernel ensure the boundedness in E of the operator

$$Kf = \int_{\Pi_+} k(x-y)f(y)\,dy, \qquad \|K\|_E \leq Q.$$

The method of solving (1) consists in seeking the Fourier coefficients of the solution $f(x_1, x_2, x_3)$ with respect to x_1, x_2. To this end we multiply (1) by

$$\frac{1}{\pi^2}\exp[-i(mx_1 + nx_2)]$$

and integrate with respect to x_1, x_2 from 0 to 2π. As a result we obtain

$$f_{mn}(x_3) = \frac{1}{4\pi^2}\int_{\Pi_+} f(y)\,dy \int_0^{2\pi}\int_0^{2\pi} k(x-y)\exp[-i(mx_1+nx_2)]\,dx_1dx_2 + g_{mn}(x_3),$$

$$f_{mn}(x_3) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x_1, x_2, x_3)\exp[-i(mx_1 + nx_2)]\,dx_1dx_2,$$

$$g_{mn}(x_3) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} g(x_1, x_2, x_3)\exp[-i(mx_1 + nx_2)]\,dx_1dx_2.$$

Introducing the notation

$$k_{mn}(u_3) = \int_0^{2\pi}\int_0^{2\pi} k(u_1, u_2, u_3)\exp[-i(mu_1 + nu_2)]\,du_1du_2 \tag{2}$$

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and using the periodicity of $k(u)$, we can write

$$f_{mn}(x_3) = \frac{1}{4\pi^2} \int_{\Pi_+} k_{mn}(x_3 - y_3) f(y) \exp[-i(my_1 + ny_2)]\, dy + g_{mn}(x_3),$$

or

$$f_{mn}(x_3) = \int_0^\infty k_{mn}(x_3 - y_3) f_{mn}(y_3)\, dy_3 + g_{mn}(x_3), \qquad (3)$$

$$m, n = 0, \pm 1, \pm 2, ...$$

Thus, to determine the Fourier coefficients of the unknown function, an infinite system of mutually independent one-dimensional equations with difference kernels has been obtained, which, in accordance with the spaces indicated above for (1), should be considered in the spaces C_0, C, M, L_p on the interval $[0, \infty]$. Since

$$Q_{mn} \equiv \int_{-\infty}^\infty |k_{mn}(u_3)|\, du_3 =$$

$$= \int_{-\infty}^\infty du_3 \left| \int_0^{2\pi} \int_0^{2\pi} k(u_1, u_2, u_3) \exp[-i(mu_1 + nu_2)]\, du_1 du_2 \right| \leq \int_\Pi |k(u)|\, du,$$

the summability conditions on the kernels reduce to (S).

Moreover, since by the Riemann-Lebesgue theorem ([6]), for almost all u_3,

$$\lim_{|m|+|n|\to\infty} \int_0^{2\pi} \int_0^{2\pi} k(u_1, u_2, u_3) \exp[-i(mu_1 + nu_2)]\, du_1 du_2 = 0,$$

then, applying Lebesgue' s theorem on passage to the limit under the integral sign, we find

$$\lim_{|m|+|n|\to\infty} Q_{mn} = 0.$$

Taking into account that the numbers Q_{mn} give upper estimates for the norms of the operators in (3), we infer from this the existence of such a positive number N that, for $|m| + |n| > N$, all equations (3) are uniquely solvable for arbitrary free terms. In particular, as N one may take a number such that $Q_{mn} < 1$ for $|m| + |n| > N$. Thus the homogeneous equation (1) is equivalent to a finite collection of homogeneous equations (3), where $|m| + |n| \leq N$. Every solution $f_{mn}(x_3)$ of the homogeneous equation (3) gives a solution of the homogeneous equation (1) by the formula
$f(x_1, x_2, x_3) = f_{mn}(x_3) \exp[i(mx_1 + nx_2)]$, which is verified by direct substitution into (1). Conversely, from the very procedure of passing from equation (1) to system (3) it follows that the Fourier coefficients $f_{mn}(x_3)$ of every solution of (1) satisfy (3). It is also clear that solutions of the homogeneous equation (1) formed from linearly independent solutions of a single equation (3), and solutions constructed from functions determined by different equations (3), are linearly independent.

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Thus, the study of the homogeneous equation (1) is completely reduced to the study of a finite collection of equations (3) for $|m| + |n| \leq N$, and this latter study is carried out on the basis of the results of (1).

As for the nonhomogeneous equation (1), here, after finding the solutions of the system (3), we must still show that they indeed form the sequence of Fourier coefficients of some function from E. We shall give an indirect proof of this fact.

Let N be the positive number chosen in the manner indicated above. Introduce for consideration the following sets of functions: E_N is the set of all functions $f(x_1, x_2, x_3) \in E$ for which the Fourier coefficients $f_{mn}(x_3)$ are equal to zero for $|m| + |n| > N$; E^N is the set of those functions $f(x_1, x_2, x_3) \in E$ for which $f_{mn}(x_3) \equiv 0$ for $|m| + |n| \leq N$.

It is not difficult to verify that E_N and E^N are closed in E, i.e., they are Banach spaces, and every element $f \in E$ is representable in the form

$$f = f_N + f^N, \qquad f_N \in E_N, \qquad f^N \in E^N,$$

and this representation is unique. Further, from the periodicity of the kernel it follows that the subspaces E_N and E^N are invariant with respect to the operator K. Therefore equation (1) can be written in the equivalent form

$$f_N = K f_N + g_N, \tag{4_1}$$

$$f^N = K f^N + g^N. \tag{4_2}$$

Here equation (4_1) is equivalent to the finite part of system (3) for $|m|+|n| \leq N$, while equation (4_2) corresponds to the remaining part of this system.

We assert that equation (4_2) is uniquely solvable in E^N for any free term. To be convinced of this, it suffices to note that:

a) each equation (3) for $|m| + |n| > N$ is uniquely solvable for arbitrary free terms;

b) the operator K can be approximated in norm by the operators

$$K_{N_1} f = \int_{\Pi_+} \sum_{|m|+|n| \leq N_1} w_{mn}(x_3 - y_3) \exp\{i[m(x_1 - y_1) + n(x_2 - y_2)]\} f(y)\, dy;$$

c) the operators K_{N_1} are annihilation operators on the subspaces E^{N_1}; therefore, on the subspaces E^{N_1}, for sufficiently large N_1, $\|K\| < 1$.

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Thus, the entire picture of the solvability of equation (1) in E is completely determined by the properties of equation (4_1) in E_N.

Bearing in mind the application to the equivalent equation (4_1) of the finite collection of equations (3) for $|m| + |n| \leq N$ and of the results from (1), we intro-

we introduce the following notation:

$$H_{mn}(t) = \int_{-\infty}^{\infty} k_{mn}(u_3) \exp[itu_3]\, du_3 = \int_{\Pi} k(u_1, u_2, u_3) \exp i(tu_3 - mu_1 - nu_2)\, du,$$

$$-\infty < t < \infty;$$

$$G_{mn}(t) = 1 - H_{mn}(t), \qquad G(t) = \prod_{|m|+|n|\leq N} [G_{mn}(t),$$

$$\varkappa_{mn} = -\operatorname{Ind}_{-\infty<t<\infty} G_{mn}(t) = -\frac{1}{2\pi}\{\arg G_{mn}(t)\}_{-\infty}^{\infty},$$

$$\varkappa_+ = \sum_{\substack{\varkappa_{mn}>0 \\ |m|+|n|\leq N}} \varkappa_{mn}, \qquad \varkappa_- = -\sum_{\substack{\varkappa_{mn}<0 \\ |m|+|n|\leq N}} \varkappa_{mn}.$$

Now, taking into account the reasoning carried out above and applying to equations (3), for $|m| + |n| \leq N$, the results from (1), we obtain the following theorems on the solvability of equation (1).

Theorem 1. Let, in equation (1), the kernel $k(u_1, u_2, u_3)$ be periodic in u_1, u_2, satisfy the summability condition (S), and let $G(t) \neq 0$, $-\infty < t < \infty$. Then:

a) the homogeneous equation has exactly $\varkappa_+$ linearly independent solutions, the same in all spaces of the series E;

b) for the solvability of the nonhomogeneous equation it is necessary and sufficient that $\varkappa_-$ solvability conditions of the form

$$\int_{\Pi_+} g(x)\psi_j(x)\, dx = 0, \qquad j = 1, 2, \ldots, \varkappa_-,$$

hold, where $\{\psi_j(x)\}$ is a basis of solutions of the homogeneous equation transposed to (1).

Theorem 2. Let $k(u_1, u_2, u_3)$ satisfy the conditions of Theorem 1. Then, if $G(t) = 0$ for at least one value of t, $-\infty < t < \infty$, the operator $I - K$ is neither a Φ- nor a $\Phi_\pm$-operator in E.

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Remark 1. Since all solutions of the transposed homogeneous equation belong to E_N, it is sufficient to check the solvability conditions for functions

$$g_N(x) = \sum_{|m|+|n|\leq N} g_{mn}(x_3)\exp[i(mx_1 + nx_2)].$$

Remark 2. Along with the spaces C_0 and C, one could also consider other subspaces of M, for example the space of functions continuous at finite points of the half-strip Π_+, the space of functions $f(x_1, x_2, x_3)$ continuous at finite points and having, as $x_3 \to \infty$, a limit independent of x_1, x_2, etc.

Remark 3. If, instead of the half-strip Π_+, one considers the strip Π, then, provided the condition $G(t) \neq 0$ is fulfilled, the integral equation will always be uniquely solvable.

Physical-Technical Institute named after S. U. Umarov
Academy of Sciences of the Tajik SSR
Dushanbe

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REFERENCES

1. M. G. Krein, *UMN*, **13**, 5 (83), 3 (1958).

2. L. S. Goldstein, I. Ts. Gokhberg, *DAN*, **131**, No. 1, 9 (1960).

3. L. S. Goldstein, *Izv. AN MSSR*, ser. phys.-math. and techn., No. 6, 27 (1964).

4. V. S. Rabinovich, in: *Theory of Functions, Functional Analysis and Their Applications*, vol. 5, Kharkov, 1967, p. 59.

5. I. B. Simonenko, *DAN*, **176**, No. 6, 1255 (1967).

6. A. Zygmund, *Trigonometric Series*, 2, Moscow, 1965.

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