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Abstract

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ON ESTIMATING THE REMAINDER OF A GENERALIZED FOURIER SERIES OF DIFFERENTIABLE FUNCTIONS OF TWO VARIABLES

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1. A. N. Kolmogorov ([1]) established the asymptotic estimate for the remainder of the Fourier series of a 2π-periodic differentiable function $f(x)$:

$$\sup_{f\in S^{(r)}} \max_x R_N(f,x) = \frac{4}{\pi^2}\frac{\ln N}{N^r} + O\left(\frac{1}{N^r}\right), \tag{1}$$

where $R_N(f,x)$ is the remainder of order N of the Fourier series of the function $f(x)$, and $S^{(r)}$ is the class of functions satisfying the condition $\left|f^{(r)}(x)\right| \le 1$. This result was subsequently generalized to broader classes of functions in a number of works by S. M. Nikol' skii (see ([2])). The transfer of some results ([1,2]) to functions of two variables is the subject of the works ([3–7]), etc. In the present note an asymptotic equality of type (1) is established in the case of approximation of functions of two variables by partial sums of the generalized Fourier series introduced by the author ([8]).

Let us recall the definition of a generalized Fourier series (with respect to trigonometric functions). Let $f(x,y)$ be a function 2π-periodic in both variables. Its generalized Fourier series is the sum of the form

$$\sum_{n=0}^{\infty} G_n(x)\cos ny + Q_n(x)\sin ny + \sum_{m=0}^{\infty} \widetilde{G}_m(y)\cos mx + \widetilde{Q}_m(y)\sin mx, \tag{2}$$

where the generalized Fourier coefficients (defined as in ([8])) have the form

$$\begin{matrix} G_n(x) \\ Q_n(x) \end{matrix} = \begin{matrix} \varphi_n(x) \\ f_n(x) \end{matrix} - \sum_{m=0}^{\infty} \frac{a_{mn}\chi_{mn}}{c_{mn}s_{mn}}\cos mx + \frac{d_{mn}t_{mn}}{b_{mn}r_{mn}}\sin mx, \tag{3}$$

$$\begin{matrix}\widetilde{G}_m(y)\\ \widetilde{Q}_m(y)\end{matrix} = \begin{matrix}\widetilde{\varphi}_m(y)\\ \tilde{f}_m(y)\end{matrix} - \sum_{n=0}^{\infty} \frac{a_{mn}(1-\chi_{mn})}{d_{mn}(1-t_{mn})} \cos ny + \frac{c_{mn}(1-s_{mn})}{b_{mn}(1-r_{mn})} \sin ny. \quad (4)$$

Here $r_{mn}, s_{mn}, t_{mn}, \chi_{mn}$ are arbitrary constants; $\varphi_n, f_n, \widetilde{\varphi}_m, \tilde{f}_m$ and $a_{mn}, b_{mn}, c_{mn}, d_{mn}$ are respectively the single and double Fourier coefficients of the function $f(x, y)$, for example

$$\begin{matrix}\varphi_n(x)\\ f_n(x)\end{matrix} = \frac{1}{\pi}\int_0^{2\pi} f(x,y) \begin{matrix}\alpha_n \cos ny\\ \sin ny\end{matrix}\, dy, \qquad \begin{matrix}a_{mn}\\ b_{mn}\end{matrix} = \frac{1}{\pi^2}\int_0^{2\pi}\int_0^{2\pi} f(x,y) \begin{matrix}\beta_{mn}\cos mx \cos ny\\ \sin mx \sin ny\end{matrix}\, dx\, dy,$$

where $\beta_{00} = 1/4$, $\alpha_0 = \beta_{0n} = \beta_{m0} = 1/2$; the remaining α_n, β_{mn} are equal to 1. It is natural to call the partial sum of the series (2) the sum

$$f_{M,N} = \sum_{n=0}^{N} G_n^{(M)}(x)\cos ny + Q_n^{(M)}(x)\sin ny + \sum_{m=0}^{M} \widetilde{G}_m^{(N)}(y)\cos mx +$$

$$+\widetilde{Q}_m^{(N)}(y)\sin mx, \quad (5)$$

where $G_n^{(M)}, Q_n^{(M)}$ and $\widetilde{G}_m^{(N)}, \widetilde{Q}_m^{(N)}$ are functions defined by the equalities (3) and (4), if the summation limits in their right-hand sides are respectively replaced–

but on M and N. It is easy to see that the series (2) and its partial sum (5) do not depend on the constants $r_{mn}, s_{mn}, t_{mn}, \chi_{mn}$. For example,

$$f_{M,N} = \sum_{n=0}^{N} \varphi_n \cos ny + f_n \sin ny + \sum_{m=0}^{M} \widetilde{\varphi}_m \cos mx + \tilde{f}_m \sin mx - \sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn}, \quad (6)$$

where

$$A_{mn} = a_{mn}\cos mx \cos ny + b_{mn}\sin mx \sin ny + c_{mn}\cos mx \sin ny + d_{mn}\sin mx \cos ny.$$

2. Here we shall consider the class $S^{(p,q)}$ (p and q are integers) of functions having almost everywhere the mixed derivative $T(x, y) = \partial^{p+q} f/\partial x^p \partial y^q$, satisfying the condition

$$|T(x, y)| \leq 1. \quad (7)$$

Theorem. *If the function $f(x,y) \in S^{(p,q)}$ and (5) is the partial sum of its generalized Fourier series, then the following asymptotic equality holds*

$$C_{M,N}^{(p,q)} \equiv \sup_{f\in S^{(p,q)}} \max_{x,y} |f - f_{M,N}| = \frac{16 \ln M \ln N}{\pi^4 M^p N^q} + O\left(\frac{\ln MN}{M^p N^q}\right). \tag{8}$$

Proof. Since $f \in S^{(p,q)}$, the expansions

$$f = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}; \qquad \varphi_n \cos ny + f_n \sin ny = \sum_{m=0}^{\infty} A_{mn}; \qquad \widetilde{\varphi}_m \cos mx +$$

$$+ \tilde{f}_m \sin mx = \sum_{n=0}^{\infty} A_{mn}.$$

Therefore, in view of (6),

$$f - f_{M,N} = \left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} - \sum_{m=0}^{M}\sum_{n=0}^{\infty} - \sum_{m=0}^{\infty}\sum_{n=0}^{N} + \sum_{m=0}^{M}\sum_{n=0}^{N}\right) A_{mn} =$$

$$= \sum_{m=M+1}^{\infty}\sum_{n=N+1}^{\infty} A_{mn}. \tag{9}$$

Since the least upper bound $C_{M,N}^{(p,q)}$ does not depend on x, y, putting $x = y = 0$ in (9), we obtain

$$\pi^2 C_{M,N}^{(p,q)} = \sup_{f\in S^{(p,q)}} \left| \int_0^{2\pi} \int_0^{2\pi} f(x,y) D_M^{(0)}(x) D_N^{(0)}(y)\, dx\, dy \right| =$$

$$= \sup_{f\in S^{(p,q)}} \left| \int_0^{2\pi} \int_0^{2\pi} T(x,y) D_M^{(p)}(x) D_N^{(q)}(y)\, dx\, dy \right|, \tag{10}$$

where

$$D_L^{(k)}(u) = \sum_{n=L+1}^{\infty} n^{-k} \cos\left(nu + \frac{k\pi}{2}\right).$$

From (10), by virtue of (7), we have

$$C_{M,N}^{(p,q)} \le \frac{1}{\pi^2} \int_0^{2\pi} |D_M^{(p)}(x)|\, dx \int_0^{2\pi} |D_N^{(q)}(y)|\, dy. \tag{11}$$

In (1) it was proved that

$$\int_0^{2\pi} |D_M^{(p)}(x)|\,dx = \frac{4}{\pi}\frac{\ln M}{M^p} + O\left(\frac{1}{M^p}\right).$$

Therefore from (11) we obtain

$$C_{M,N}^{(p,q)} \le \frac{16}{\pi^4}\frac{\ln M \ln N}{M^p N^q} + O\left(\frac{\ln MN}{M^p N^q}\right). \tag{12}$$

To complete the proof of the theorem it remains to determine a function $T(x,y)$, satisfying condition (7) and the condition

$$\int_0^{2\pi} T(x,y)\,dx = \int_0^{2\pi} T(x,y)\,dy = 0,$$

such that in relation (12) the equality sign would hold. Put $T = X(x)Y(y)$, where, for even and odd p, respectively,

$$X(x) = \begin{cases} +1, & \text{if } \sin\dfrac{2M+1}{2}x > 0, \quad x > \dfrac{2\pi}{2M+1}, \\ 0, & \text{if } 0 \le x \le 2\pi/(2M+1), \\ -1, & \text{if } \sin\dfrac{2M+1}{2}x < 0, \quad x > \dfrac{2\pi}{2M+1}; \end{cases}$$

$$X(x) = \begin{cases} +1, & \text{if } D_M^{(p)}(x) > 0, \\ -1, & \text{if } D_M^{(p)}(x) < 0. \end{cases}$$

The function $Y(y)$ is defined analogously for even and odd q. Next, in exactly the same way as in (1), it is shown (taking into account the separation of variables in the integral (10)) that for the function $T(x,y)$ thus defined the estimate (12) is attained. The theorem is proved.

Remark 1. Let us compare estimate (8) with the estimate of the remainders of ordinary Fourier series. It is known that

$$\sup_{f\in S^{(p,q)}} \max_{x,y} \left| f - \sum_{m=0}^{M}\sum_{n=0}^{N} A_{mn} \right| = O\left(\frac{\ln M}{M^p} + \frac{\ln M}{N^q}\right), \tag{13}$$

$$\sup_{f\in S^{(p,q)}} \max_{x,y} \left| f - \sum_{n=0}^{M+N+1} \varphi_n \cos ny + \bar{f}_n \sin ny \right| = O\left[\frac{\ln(M+N)}{(M+N)^q}\right]. \tag{14}$$

(We have taken in (14) a partial sum of the ordinary Fourier series of order $M+N+1$, so that the number of its terms would be no smaller than the number of terms of the partial sum (5).) Obviously, estimate (8) is substantially better than estimates (13) and (14), if M and N (or p and q) are numbers of the same order.

Remark 2. In an analogous way estimates of type (8) are established in the case of other classes of functions. Thus, for example,

$$\sup_{f\in S^{(p,q)}H^{(\alpha,\beta)}} \max_{x,y} |f - f_{M,N}| = \frac{8}{\pi^4}\frac{\ln M \ln N}{M^p N^q}\int_0^{\pi/2}\int_0^{\pi/2} \min\left[K_1\left(\frac{2u}{M}\right)^{\alpha}, K_2\left(\frac{2\vartheta}{N}\right)^{\beta}\right]\sin u \sin\vartheta \, du\, d\vartheta + O\left[\frac{(M^-}{}\right.$$

$$\sup_{f\in S^{(p,q)}H_*^{(\alpha,\beta)}} \max_{x,y} |f - f_{M,N}| = \frac{4K}{\pi^4}\frac{\ln M \ln N}{M^{p+\alpha}N^{q+\beta}}\int_0^{\pi/2}(2u)^{\alpha}\sin u\, du \int_0^{\pi/2}(2\vartheta)^{\beta}\sin\vartheta\, d\vartheta + O\left(\frac{\ln MN}{M^{p+\alpha}N^{q+\beta}}\right).$$

Here $S^{(p,q)}H^{(\alpha,\beta)}$ and $S^{(p,q)}H_*^{(\alpha,\beta)}$ denote classes of functions $f(x,y)$ whose mixed derivative $T(x,y)$ satisfies, respectively, the conditions $(0 < \alpha, \beta < 1)$

$$|T(x_1,y_1) - T(x_2,y_2)| \le K_1|x_1 - x_2|^{\alpha} + K_2|y_1 - y_2|^{\beta},$$

$$|T(x_1,y_1) - T(x_1,y_2) - T(x_2,y_1) + T(x_2,y_2)| \le K|x_1 - x_2|^{\alpha}|y_1 - y_2|^{\beta}.$$

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Note: Figure translations are in progress. See original paper for figures.