



---

Soviet-era science, translated into English

# ON A PRIORI ESTIMATES

MATHEMATICS

1969

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.27488>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 517.43

*MATHEMATICS*

Yu. V. EGOROV

## ON A PRIORI ESTIMATES

## FOR FIRST-ORDER DIFFERENTIAL OPERATORS

*(Presented by Academician I. G. Petrovskii on 18 VII 1968)*

In the present work exact necessary and sufficient conditions are obtained under which estimates of the form

$$\int |u(x)|^2 e^{Q(x,\mu)} dx \leq C \int \left| \sum_1^n a_j \frac{\partial u}{\partial x_j} \right|^2 e^{Q(x,\mu)} dx, \quad u \in C_0^\infty(\mathbf{R}^n), \quad (1)$$

hold, where  $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ ;  $\mu = (\mu_1, \dots, \mu_r)$ ,  $\mu_j \geq 0$ ;  $(a_1, \dots, a_n)$  is a constant vector from  $C^n$ ;  $Q(x, \mu)$  is a homogeneous real polynomial in  $(x_1, \dots, x_n, \mu)$  of degree  $k$ ; the constant  $C$  does not depend on  $u$  and  $\mu$ . The infinitely differentiable scalar function  $u(x)$  takes complex values.

Inequalities of this type under various assumptions on  $Q$  were studied in papers (1-4) and others. In Chap. IV of the book (4) it is shown how, by means of similar inequalities, existence theorems are obtained for a solution of the equation  $\bar{\partial}u = f$  in the whole space.

However, in all the works known to us only the case  $k = 2$  has been completely investigated. In papers (5, 6) we studied the case  $k = 3$ . From these papers it is also clear how inequalities of type (1) are used to study pseudodifferential operators of principal type. For example, using the results of the present work one can show that for the operator  $P$  with symbol whose principal part is

$$p^0(x, \xi) = i\xi_1 + ax_1^k |\xi|,$$

the estimate

$$|u|_{1/(k+1)} \leq C(K)(|Pu|_0 + |u|_0), \quad u \in C_0^\infty(K) \quad (2)$$

holds ( $K$  is an arbitrary compact set in  $\mathbf{R}^n$ ) if and only if  $k$  is even and  $a \neq 0$ , or if  $k$  is odd and  $a < 0$  (see (7)). Such an operator arises in solving the oblique-derivative problem (see (8)).

Similarly, for the operator  $P$  with symbol whose principal part is equal to

$$p^0(x, \xi) = i\xi_1 + \xi_2 + ax_1^k|\xi|,$$

estimate (2) holds if and only if  $k$  is odd and  $a < 0$ .

We note that estimate (1) is equivalent to the estimate

$$\int |u(x)|^2 e^{tQ(x, \mu)} dx \leq Ct^{-1/k} \int \left| \sum_1^n a_j \frac{\partial u}{\partial x_j} \right|^2 e^{tQ(x, \mu)} dx, \quad u \in C_0^\infty(M_1), \quad (3)$$

where  $M_1$  is the ball  $\{x : \sum_1^n x_j^2 \leq 1\}$ ;  $t$  is a parameter varying on the half-line  $1 \leq t < \infty$ ; the constant  $C$  does not depend on  $u$ ,  $\mu$ , and  $t$ . Estimate (3) is obtained from (1) after replacing  $x$  by  $xt^{1/k}$ , and  $\mu$  by  $\mu t^{1/k}$ .

On the other hand, the possibility of the substitution  $u(x) = v(x)e^{-\frac{1}{2}Q(x, \mu)}$  means that estimate (1) is equivalent to an inequality of the form

$$\int |v(x)|^2 dx \leq C \int \left| \sum_{j=1}^n a_j \frac{\partial v}{\partial x_j} + P(x, \mu)v \right|^2 dx, \quad v \in C_0^\infty(\mathbf{R}^n),$$

where  $P(x, \mu)$  is a homogeneous polynomial of degree  $k - 1$  with complex coefficients.

**Theorem 1.** If the vector  $(a_1, \dots, a_n) \equiv a$  is real, then estimate (1) holds if and only if the function  $Q(x + at, \mu)$  of the real variable  $t$ ,  $-\infty < t < +\infty$ , for any fixed values of  $(x_1, \dots, x_n, \mu)$ , is not identically a constant and has no points of local maximum.

**Theorem 2.** If the vector  $(a_1, \dots, a_n)$  is not proportional to a real vector, estimate (1) holds if and only if the function of the real variables  $s$  and  $t$ ,  $-\infty < s, t < +\infty$ ,

$$\sum_1^n \frac{\partial^2 Q}{\partial x_j \partial x_k}(x + t \operatorname{Re} a + s \operatorname{Im} a, \mu) a_j \bar{a}_k \quad \text{is nonnegative} \quad (4)$$

and is not identically zero for any fixed values of  $(x_1, \dots, x_n, \mu)$ .

In particular, it follows from Theorem 2 that estimate (1) cannot hold for odd  $k$  if the vector  $(a_1, \dots, a_n)$  is not proportional to a real one. We note that in the

case  $k = 2$  our conditions coincide with the conditions of Lemma 8.1.3 in the book <sup>(3)</sup>.

The proof of Theorem 1 is based on rather simple, but nontrivial, considerations concerning the theory of functions of a real variable. The proof of Theorem 2 is more complicated. To prove the necessity of condition (4), we show that inequality (1) implies an estimate for a certain pseudodifferential operator in the domain  $M_1$ , and condition (4) is necessary for this estimate. The proof of sufficiency uses, in particular, the technique of repeated commutators, developed by Hörmander in <sup>(8)</sup>. An essential point in our proof is the Campbell-Hausdorff formula.

We hope that the results of this work will make it possible to obtain exact algebraic conditions on the symbol of an operator (and its derivatives) determining the operators for which an estimate of the form

$$|u|_s \leq C(K)(|Pu|_{s-m+\delta} + |u|_{s-1}), \quad u \in C_0^\infty(K), \quad 0 < \delta < 1,$$

holds, where  $K$  is a compact set and  $m$  is the order of the operator  $P$  (see <sup>(5-7)</sup>).

Moscow State University  
named after M. V. Lomonosov

Received  
2 VII 1968

## REFERENCES

- <sup>1</sup> F. Trèves, *Acta Math.*, **101**, 1 (1959).
- <sup>2</sup> B. Malgrange, *Math. Zs.*, **72**, 184 (1959).
- <sup>3</sup> L. Hörmander, *Linear Differential Operators with Partial Derivatives*, Moscow, 1965.
- <sup>4</sup> L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, Moscow, 1968.
- <sup>5</sup> Yu. V. Egorov, DAN, **182**, No. 6 (1968).
- <sup>6</sup> Yu. V. Egorov, *Matem. sborn.*, **80** (122), No. 1 (1969).
- <sup>7</sup> Yu. V. Egorov, *Matem. sborn.*, **73** (115), No. 3, 356 (1967).
- <sup>8</sup> Yu. V. Egorov, V. A. Kondrat'ev, DAN, **170**, No. 4, 770 (1966).
- <sup>9</sup> L. Hörmander, *Acta math.*, **119**, 147 (1968).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*