ON TOPOLOGICAL EQUIVALENTS OF TYCHONOFF’ S THEOREM

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Abstract

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MATHEMATICS

I. I. PAROVICHENKO

ON TOPOLOGICAL EQUIVALENTS OF TYCHONOFF' S THEOREM

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1. According to the well-known theorem of Kelley, Tychonoff' s theorem on the bicompactness of the product of bicompact T_1-spaces is equivalent to the axiom of choice (AC). In ([2]), Uok showed that, assuming the weak axiom of choice (ACF) (choice for a family of nonempty finite sets), the following three propositions are equivalent:

(TF). Tychonoff' s theorem for the product of finite spaces.

(R). (Rado' s theorem). Let $\{X_i \mid i \in I\}$ be a family of nonempty finite sets. Suppose, further, that for each finite subset A of the set I a choice function f_A is given for the family $\{X_i \mid i \in A\}$. Then there exists a choice function f for $\{X_i \mid i \in I\}$ such that for any finite subset A of the set I there exists a finite set B such that $A \subseteq B \subseteq I$ and $f(i) = f_B(i)$ for all $i \in A$.

(AL). Alexander' s closed subbase lemma for bicompactness.

Let us add the following propositions:

(TT_2). Tychonoff' s theorem for the product of bicompact T_2-spaces.

(V). Vietoris' theorem on the bicompactness of the exponential space over a bicompact space.

The main purpose of the present paper is the following

Theorem. *Without assuming the weak axiom of choice, the propositions* (TF)*,* (AL)*, and* (V) *are equivalent to Tychonoff' s theorem for bicompact T_2-spaces* (TT_2).

2. Let us add three more propositions:

(R_2^+). Let a set I be given, and for each of its finite subsets A let a nonempty family $F(A)$ of functions be given, taking the values 0 and 1, whose domains of definition lie in I and contain A, with

$$A_1 \subseteq A_2 \Rightarrow F(A_1) \supseteq F(A_2).$$

Then on I there exists a function f such that for any finite subset A of the set I there exists $\varphi \in F(A)$ such that $\varphi(i) = f(i)$ for all $i \in A$.

(TD). The Tychonoff product of two-point spaces $\Pi\{D_i \mid i \in I\}$ is bicompact.

(Φ). (the maximal-filter principle). Every filter in the set of all subsets of a set X is contained in a maximal filter.

The proof of the theorem is carried out according to the scheme:

$$(\mathrm{AL}) \Rightarrow (\mathrm{V}) \Rightarrow (\mathrm{TT}_2) \Rightarrow (\mathrm{TF}) \Rightarrow (\mathrm{TD}) \Rightarrow (\mathrm{R}_2^+) \Rightarrow (\Phi) \Rightarrow (\mathrm{AL}).$$

The first two implications follow from the remark at the end of our preceding paper ([1]). $(\mathrm{TT}_2) \Rightarrow (\mathrm{TF}) \Rightarrow (\mathrm{TD})$ is obvious. $(\mathrm{TD}) \Rightarrow (\mathrm{R}_2^+)$ is proved analogously to Gottschalk' s implication ([2]): it is enough to consider the (centered) system of closed sets Φ_A of the product ΠD_i, corresponding to finite $A \subset I$, which are defined as follows. First one takes the restrictions of all functions in $F(A)$ to the set A (there are finitely many such restrictions!), after which Φ_A is taken to be all possible extensions of the functions so obtained from A to all of I. Since

the proof $(\Phi) \Rightarrow (\mathrm{AL})$ is generally known, so it remains only to show that $(\mathrm{R}_2^+) \Rightarrow (\Phi)$, which we shall do.

3. As is known, a **two-valued measure** on a Boolean algebra B is a nonconstant mapping $B \to \{0, 1\}$ satisfying finite additivity for disjoint elements. At the same time, two-valued measures on B are nothing other than the characteristic functions of maximal filters in B. If Σ is a filter in B, then we shall call a subalgebra B_0 of B **Σ-admissible** if it contains a maximal (relative to B_0) filter Σ^+ containing Σ. If Σ is a filter in B, then by $I(\Sigma)$ we denote the ideal of all complements of elements of Σ. Obviously, $\Sigma \cup I(\Sigma)$ is a Σ-admissible subalgebra of B. Below $P(X)$ denotes the algebra of all subsets of X.

Lemma. *If Σ is a filter in $P(X)$ and A is a finite subset of $P(X)$, then there exists a Σ-admissible subalgebra in $P(X)$ containing A.*

It suffices to consider the case where A consists of a single set: $A = \{M_1\}$, since the remainder can be proved by induction. If $M_1 \in \Sigma \cup I(\Sigma)$, then there is nothing to prove, so let this not be so. Then M_1 intersects every element of Σ (otherwise $M_1 \in I(\Sigma)$). Adding all these intersections, as well as all their supersets, to Σ, we obtain a filter $\Sigma^+ \supseteq \Sigma$, which will be maximal in the subalgebra $\Sigma^+ \cup I(\Sigma^+)$, containing M_1.

$(\mathrm{R}_2^+) \Rightarrow (\Phi)$. Let Σ be a filter in $P(X)$; put, for (R_2^+), $I = P(X)$, and take as $F(A)$ all two-valued measures on all Σ-admissible subalgebras containing A. The lemma then ensures that $F(A)$ is nonempty. Obviously, f will be a measure on $P(X)$, and $\{M \mid f(M) = 1\}$ is a maximal filter in $P(X)$ containing Σ.

In conclusion we note that in Wolk's paper ([2]), in proving the equivalence of (TF) and (AL), the use of the weak axiom of choice is essential in both implications needed for this, which it was possible to avoid in the present work. Since the list of effective equivalents in our sense also includes the maximal filter principle, this list can be substantially enlarged by many other assertions from topology, set theory, and mathematical logic. In particular, the equivalence of (TT_2) and (Φ), which we have proved here incidentally, has already been mentioned in the literature. And, finally, we note that the conjunction (R & ACF) can be added to our list, since (TR) $\Rightarrow$ (R & ACF) $\Rightarrow$ (Φ).

Kishinev State
University

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Note: Figure translations are in progress. See original paper for figures.