Reports of the Academy of Sciences of the USSR

MATHEMATICS

1969

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1969. Volume 189, No. 3

UDC 517.544.3

MATHEMATICS

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A BOUNDARY-VALUE PROBLEM FOR AN OVERDETERMINED SYSTEM OF TWO SECOND-ORDER EQUATIONS

(Presented by Academician A. Yu. Ishlinskii on 8 IV 1968)

1. Let an overdetermined system be given

$$P_1(\partial/\partial x)u = 0, \qquad P_2(\partial/\partial x)u = 0, \tag{1}$$

where $x = (x_1, x_2, x_3)$ is a point of three-dimensional space, $\partial/\partial x = (\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$, and $P_1(\partial/\partial x)$ and $P_2(\partial/\partial x)$ are homogeneous second-order differential operators with constant real coefficients. The system is called elliptic (see ([1])) if

$$|P_1(\xi)|^2 + P_2(\xi)^2 > 0 \quad \text{for } \xi \neq 0. \tag{2}$$

Let $C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ be an arbitrary nonsingular matrix. The system

$$c_{11}P_1u + c_{12}P_2u = 0, \qquad c_{21}P_1u + c_{22}P_2u = 0 \tag{3}$$

is equivalent to system (1).

Using such a transformation and a linear change of variables, we shall reduce system (1) to canonical form.

Lemma 1. *If condition (2) is satisfied, there exist such λ and μ that the operator $P = \lambda P_1 + \mu P_2$ is elliptic.*

Lemma 2. *There exists a coordinate system $y = (y_1, y_2, y_3)$ in which system (1) is equivalent to one of the following systems of equations*

$$\partial^2 u/\partial y_1^2 + \partial^2 u/\partial y_2^2 = 0, \qquad \partial^2 u/\partial y_2^2 + \partial^2 u/\partial y_3^2 = 0 \tag{4}$$

or

$$\partial^2 u/\partial y_1^2 + \partial^2 u/\partial y_2^2 = 0, \qquad \partial^2 u/\partial y_3^2 = 0. \tag{5}$$

The proof of Lemma 2 follows from Lemma 1 and from the possibility of simultaneously reducing two quadratic forms to diagonal form.

2. Consider the following boundary-value problem for system (4) in the trihedral angle $y_1 > 0,\ y_2 > 0,\ y_3 > 0$:

$$u(y_1, 0, 0) = f_1(y), \qquad u(0, y_2, 0) = f_2(y_2),$$

$$u(0, 0, y_3) = f_3(y_3), \qquad \frac{\partial u}{\partial y_1}(0, 0, y_3) = f_4(y_3), \tag{6}$$

where $f_i(t)$ are sufficiently smooth and decreasing functions, $1 \le i \le 4$.

Theorem 1. *The solution of the boundary-value problem (6) for system (4) is unique and exists under a finite number of conditions on f_i, $1 \le i \le 4$.*

We seek the solution of the boundary-value problem (6) for system (4) in the form $u = \sum_{i=1}^{3} u_i$, where u_i are solutions in the corresponding dihedral angles whose intersections form the trihedral angle. The solution of the system in a dihedral angle is easily written explicitly by means of the Fourier transform.

Let u_1 be a solution of system (4) in the dihedral angle $y_1 > 0,\ y_3 > 0$. We make the Fourier transform with respect to y_2. We obtain the system of ordinary differential equations

$$\partial^2 \tilde{u}_1/\partial y_1^2 - \xi_2^2 \tilde{u}_1 = 0, \qquad \partial^2 \tilde{u}_1/\partial y_3^2 - \xi_2^2 \tilde{u}_1 = 0.$$

The general solution of this system, bounded for $y_1 > 0,\ y_3 > 0$, has the form

$$\tilde{u}_1 = \tilde{g}_1(\xi_2) \exp(-|\xi_2| y_1 - |\xi_2| y_3).$$

Making the inverse Fourier transform with respect to ξ_2, we obtain

$$\begin{aligned} u_1(x_1, x_2, x_3) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{g}_1(\xi_2) \exp(-y_1|\xi_2| - y_3|\xi_2| - i y_2 \xi_2)\, d\xi_2 \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(y_1 + y_3) g_1(t)\, dt}{(y_1 + y_3)^2 + (y_2 - t)^2}, \end{aligned} \tag{7}$$

where $g_1(t)$ is an arbitrary density. In what follows we shall assume that $g_1(t) = 0$ for $t < 0$.

Let u_2 be a solution of system (4) in the dihedral angle $y_2 > 0,\ y_3 > 0$. Making the Fourier transform with respect to y_1 and subtracting the second equation in (4) from the first, we obtain the system of ordinary differential equations

$$\partial^2 \tilde{u}_2/\partial y_2^2 - \xi_1^2 \tilde{u}_2 = 0, \qquad \partial^2 \tilde{u}_2/\partial y_3^2 + \xi_1^2 \tilde{u}_2 = 0.$$

The general solution of this system, bounded for $y_2 > 0,\ y_3 > 0$, has the form

$$\tilde{u}_2 = \tilde{g}_2(\xi_1)\exp(-|\xi_1|y_2 + i\xi_1 y_3) + \tilde{g}_3(\xi_1)\exp(-|\xi_1|y_2 - i\xi_1 y_3).$$

Making the inverse Fourier transform with respect to ξ_1, we obtain

$$u_2 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y_2 g_2(t)\,dt}{y_2^2 + (y_1 - y_3 - t)^2} + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y_2 g_3(t)\,dt}{y_2^2 + (y_1 + y_3 - t)^2}, \tag{8}$$

where $g_2(t)$ and $g_3(t)$ are arbitrary densities, which we shall assume to be zero for $t < 0$.

Similarly, the solution u_3 in the dihedral angle $y_1 > 0,\ y_2 > 0$ is written in the form

$$u_3 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y_2 g_4(t)\,dt}{y_2^2 + (y_3 - y_1 - t)^2} + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y_2 g_5(t)\,dt}{y_2^2 + (y_3 + y_1 - t)^2}. \tag{9}$$

We note that the kernels of the second terms in (8) and in (9) are the same; therefore we seek the solution of boundary-value problem (6) for system (4) in the form

$$\begin{aligned} u(y) = {} & \frac{1}{\pi}\int_0^{\infty} \frac{(y_1 + y_3) g_1(t)\,dt}{(y_1 + y_3)^2 (y_2 - t)^2} + \frac{1}{\pi}\int_0^{\infty} \frac{y_2 g_2(t)\,dt}{y_2^2 + (y_1 - y_3 - t)^2} \\ & + \frac{1}{\pi}\int_0^{\infty} \frac{y_2 g_3(t)\,dt}{y_2^2 + (y_1 + y_3 - t)^2} + \frac{1}{\pi}\int_0^{\infty} \frac{y_2 g_4(t)\,dt}{y_2^2 + (y_3 - y_1 - t)^2}. \end{aligned} \tag{10}$$

Substituting expression (10) into the boundary conditions (6) and assuming $g_i(0) = 0,\ 1 \le i \le 4$, we obtain a system of 4 integro-differential equations for determining the unknown densities $g_i(t),\ 1 \le i \le 4$. Application of the Mellin transform (see ([2,3])) reduces this system to an algebraic one, whence follows the existence of a sufficiently smooth solution of boundary-value problem (6) for system (4), provided only that $f_i,\ 1 \le i \le 4$, satisfy a finite number of conditions. It is not difficult to verify that the solution of boundary-value problem (6) for system (4) is unique.

Let us note that, analogously to problem (6), one can consider problem (6′) with boundary conditions

$$u(y_1,0,0)=f_1(y_1), \qquad u(0,y_2,0)=f_2(y_2), \tag{6'}$$

$$u(0,0,y_3)=f_3(y_3), \qquad \partial u/\partial y_3(y_1,0,0)=f_4(y_1).$$

3. Let $ABCDA'B'C'D'$ be a parallelepiped with edges parallel to the coordinate axes (see Fig. 1). In this case the boundary conditions are prescribed as follows: on the edges AA' and CC' one must prescribe the values of $u(y)$ and $\partial u/\partial y_1$; on the edges BB' and DD' no boundary condition is prescribed, and on all the remaining edges the value $u(y)$ is prescribed. Thus, the boundary conditions have the form

$$\partial u/\partial y_1\big|_{AA'+CC'}=f_1(t), \qquad u\big|_{AB+BC+CD+DA}=f_2(t), \tag{11}$$

$$u\big|_{AA'+CC'}=f_3(t), \qquad u\big|_{A'B'+B'C'+C'D'+D'A'}=f_4(t).$$

Fig. 1

The functions $f_i(t)$ are assumed to be sufficiently smooth on each edge.

Theorem 2. *The solution of boundary-value problem (11) for system (4) in the class H_s ($s \geqslant 2$) is unique and exists provided a finite number of conditions on f_i, $1 \leqslant i \leqslant 4$, are satisfied.*

We prove that if in the boundary conditions (11) $f_i=0$, $i=1,\dots,4$, then the solution of system (4) is identically equal to zero in the parallelepiped. Consider $u(y)$ on the face $AA'D'D$. We obtain a mixed problem for a hyperbolic equation in a rectangle; hence $u(y)$ on this face is equal to zero. Similarly we obtain that $u(y)$ is equal to zero on $BB'C'C$. On the faces $ABCD$ and $A'B'C'D'$ we have the Dirichlet problem for the Laplace equation in a rectangle; hence on these faces as well the solution is equal to zero. Now take any point in the parallelepiped and pass through it a plane parallel to $A'B'BA$. In this section we obtain the Dirichlet problem for the Laplace equation in a rectangle, with the value of $u(y)$ on the boundary of the rectangle equal to zero. Consequently, the solution is equal to zero in this section.

The proof of existence is carried out analogously to the case of a trihedral angle. The solution of boundary-value problem (11) is sought in the form of a sum of solutions of system (4) in dihedral angles formed by the faces of the parallelepiped $ABCDA'B'C'D'$. Substituting this solution into the boundary conditions (11), we obtain a system of integro-differential equations for the unknown densities. With the aid of the Mellin transform this system can be reduced to Fredholm equations of the second kind, whence follows the solvability of the boundary-value problem provided a finite number of conditions on the right-hand sides of the boundary conditions are satisfied.

4. Let us briefly consider a simpler case, when system (1) reduces to system (5). Let system (5) also be considered in the parallelepiped $ABCDA'B'C'D'$. The general solution of system (5) has the form

$$u(y) = \varphi_1(y_1, y_2)y_3 + \varphi_2(y_1, y_2),$$

where φ_1 and φ_2 are arbitrary harmonic functions of two variables. Consequently, the boundary conditions for system (5) may be prescribed as follows:

$$u\big|_{AB+BC+CD+DA} = f_1(t), \qquad u\big|_{A'B'+B'C'+C'D'+D'A'} = f_2(t) \tag{12}$$

or else

$$u\big|_{AB+BC+CD+DA} = f_1(t), \qquad \partial u/\partial y_3\big|_{AB+BC+CD+DA} = f_2(t). \tag{13}$$

It is obvious that the solution of boundary-value problem (12) for system (5), as well as the solution of boundary-value problem (13) for system (5), exists and is unique.

5. We now consider the more general case, when system (1) is considered in a convex polyhedron M of fairly general form.

Definition. We shall call the face x_1x_2 of the dihedral angle $x_1 > 0,\ x_3 > 0$ **elliptic** if the system

$$P_1(-i\xi_1, -i\xi_2, \lambda) = 0, \quad P_2(-i\xi_1, -i\xi_2, \lambda) = 0 \tag{14}$$

has no nonzero solutions for ξ_1 and ξ_2 real and not simultaneously equal to zero. From the fact that both faces are elliptic it follows that, if λ_1 and λ_2 satisfy the system

$$P_1(\lambda_1, -i\xi_2, \lambda_2) = 0, \quad P_2(\lambda_1, -i\xi_2, \lambda_2) = 0, \tag{15}$$

then $\operatorname{Re}\lambda_1 \neq 0$ and $\operatorname{Re}\lambda_2 \neq 0$.

Three cases may occur: a) among the solutions of system (15) there is no pair with negative real parts; b) among the solutions there is one pair for which $\operatorname{Re}\lambda_1 \neq 0$ and $\operatorname{Re}\lambda_2 \neq 0$; c) among the solutions there are two pairs with negative real parts. The first case corresponds to the fact that no boundary conditions need be prescribed on the edge of the dihedral angle. In the second case one boundary condition must be prescribed on the edge of the dihedral angle, and in the third case two boundary conditions. In the second case, the boundary operator $B(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ is subject to a condition of Shapiro–Lopatinskii type

$$B(\lambda_1, -i\xi_2, \lambda_2) \neq 0 \quad \text{for } \xi_2 \neq 0. \tag{16}$$

In the third case, when system (15) has two pairs of solutions λ_1, λ_2 and μ_1, μ_2 with negative real parts, the condition of Shapiro–Lopatinskii type imposed on the boundary operators B_1 and B_2 has the form

$$\det \begin{Vmatrix} B_1(\lambda_1, -i\xi_2, \lambda_2) & B_1(\mu_1, -i\xi_2, \mu_2) \\ B_2(\lambda_1, -i\xi_2, \lambda_2) & B_2(\mu_1, -i\xi_2, \mu_2) \end{Vmatrix} \neq 0 \quad \text{for } \xi_2 \neq 0. \tag{17}$$

Let

$$\Gamma = \bigcup_{i=1}^{n} \Gamma_i$$

be the skeleton of the polyhedron M. Suppose that each face of M is elliptic. Let the edges Γ_i be numbered so that on Γ_i, for $1 \leq i \leq n_1$, one boundary condition must be prescribed; for $n_1 + 1 \leq i \leq n_1 + n_2$, two boundary conditions must be prescribed; and for $n_1 + n_2 + 1 \leq i \leq n_1 + n_2 + n_3$, nothing need be prescribed. Obviously, $n = n_1 + n_2 + n_3$. The boundary-value problem for system (1) in the polyhedron M has the form

$$B_i u\big|_{\Gamma_i} = h_i(t), \quad 1 \leq i \leq n_1 + n_2; \tag{18}$$

$$B_{i+n_2} u\big|_{\Gamma_i} = h_{i+n_2}(t), \quad n_1 + 1 \leq i \leq n_1 + n_2, \tag{19}$$

where B_i are differential operators of order m_i, $1 \leq i \leq n_1 + 2n_2$.

Theorem 3. Let system (1) be elliptic. Let M be a convex polyhedron with elliptic faces. Require that the operators (18), (19) satisfy the Shapiro–Lopatinskii condition of the form (16), (17). Let $m_0 = \max m_i$, $1 \leq i \leq n_1 + 2n_2$, and let s be an integer, $s \geq m_0 + 1$. Suppose that $h_i(t) \in H_{s-m_i-1}$. Then, for almost all polyhedra M with the properties indicated above, and upon fulfillment of a finite number of conditions on $h_i(t)$, there exists a solution of the boundary-value problem (18), (19) for system (1).

The boundary-value problem (18), (19) for system (1) has a finite number of linearly independent solutions in $H_s(M)$.

The proof of existence is carried out analogously to the case of a parallelepiped. An a priori estimate in H_s is also proved for the solution of the boundary-value problem (1), (18), (19), whence it follows that there exists only a finite number of linearly independent solutions of the homogeneous system.

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Received
4 III 1968

REFERENCES

1. V. P. Palamodov, *Linear differential operators with constant coefficients*, Nauka, 1967.

2. G. Titchmarsh, *Introduction to the theory of Fourier integrals*, Moscow, 1948.

3. G. I. Eskin, UMN, 18, no. 3 (1963).

Note: Figure translations are in progress. See original paper for figures.