ON INDUCTIVE DIMENSIONS

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Abstract

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All spaces considered are assumed to be normal (or, if this is specially stipulated, completely regular).

I. The aim of this part of the note is to establish conditions under which the inequalities

$$\operatorname{Ind} X_1 \times \dots \times X_s \leq \operatorname{Ind} X_1 + \dots + \operatorname{Ind} X_s, \tag{*}$$

$$\operatorname{ind} X_1 \times \dots \times X_s \leq \operatorname{ind} X_1 + \dots + \operatorname{ind} X_s. \tag{**}$$

are valid.

In works devoted to inequality (∗), an extremely restrictive and difficult-to-verify condition of total normality (³) was usually imposed on the product $X_1 \times \dots \times X_s$ outside the class of metric spaces (²), ensuring that in $X_1 \times \dots \times X_s$ the sum theorem for the dimension Ind holds.

The first genuinely sufficiently general and simple conditions for the validity of inequality (∗) were obtained by I. K. Lifanov in (¹). In particular, he showed that inequality (∗) for bicompacts already holds when not the whole product $X_1 \times \dots \times X_s$, but only the factors X_i, are totally normal. At the basis of I. K. Lifanov' s proof lies a very interesting idea: for the validity of inequality (∗), the sum theorem in $X_1 \times \dots \times X_s$ for the dimension Ind need not hold for all closed subsets of $X_1 \times \dots \times X_s$, but only for "rectangles," which is ensured (in the case of bicompacts) by imposing conditions only on the factors, and not on the whole product.

I. K. Lifanov' s work served as the stimulus for obtaining the results of the first part of this note.

We shall say that in the space X the finite sum theorem holds for the dimension Ind, respectively ind, if

$$\operatorname{Ind} F_1 \cup F_2 == \max(\operatorname{Ind} F_1, \operatorname{Ind} F_2),$$

respectively

$$\operatorname{ind} F_1 \cup F_2 = \max(\operatorname{ind} F_1, \operatorname{ind} F_2),$$

for any two boundaries F_1 and F_2 of open subsets in X.*

Theorem 1. *If the product $X_1 \times X_2$ is paracompact and is an F-product (^{4}), and in the spaces X_1 and X_2 the finite sum theorem holds for the dimension* Ind*, then formula* $(*)$ *is valid (for $s = 2$).*

Corollary 1. *If in the almost metrizable spaces X_i,** $i = 1, \dots, s$, the finite sum theorem holds for the dimension Ind, then formula $(*)$ is valid.*

Corollary 2. *If in paracompacts that are complete in the sense of Čech (in particular, in locally bicompact paracompacts, for example in bicompacts),* X_i, $i = 1, \dots, s$, the finite sum theorem for the dimension Ind holds, then formula $(*)$ is valid.

Corollary 3. Formula $(*)$ is valid if, in the factors X_i, the finite sum theorem holds for the dimension Ind, all the spaces X_i are paracompact, and all the spaces X_i, except possibly one, are locally bicompact.

Obviously, Corollaries 1 and 3 generalize Corollaries 1 and 3 from (4) and the theorem from (1).

Theorem 2. The inequality $(**)$ is valid if, in the factors X_i, the finite sum theorem holds for the dimension ind (the spaces X_i are completely regular).

The results formulated follow from consideration of the properties of a new dimensional invariant:

Definition 1. We put $\operatorname{Ind} \Lambda = -1$ and $\operatorname{Ind} X \leq n$, $n = -1, 0, 1, 2, \dots$, if in X there exist systems σ_i, $i = -1, 0, 1, \dots, n$, of subsets closed in X, satisfying the conditions: 1) the system σ_{-1} consists only of the empty set, $\sigma_n \ni X$, $\sigma_i \subseteq \sigma_{i+1}$, $i = -1, 0, 1, \dots, n-1$; 2) for any two nonempty closed disjoint subsets Φ and Ψ of an element F of the system σ_i, $i = 0, 1, \dots, n$, there is an element of the system σ_{i-1} separating Φ from Ψ in F; 3) for any two elements

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* The fact that it is appropriate to require the sum theorem to hold not for arbitrary closed sets, but only for boundaries of open sets, was first noted by I. K. Lifanov (^{1}).

** By an almost metrizable space I mean a space admitting a perfect mapping onto a metric space. As shown in (^{5}), such spaces admit arbitrarily fine (in the sense of ω-mappings) perfect mappings onto metric spaces. It is shown in (^{6}) that the class of almost metrizable spaces coincides with the class of feathered paracompacts (p-paracompacts). In (^{4}) it is proposed to call almost metrizable spaces M'-spaces, for the class of almost metrizable spaces coincides with the class of paracompact M-spaces.

F' and F'' of the system σ_i, $i = 0, 1, \dots, n$, there exists an element $F \in \sigma_i$ containing the sum $F' \cup F''$.

If in item 2) of Definition 1 the pair of closed sets Φ and Ψ is replaced by a closed set and a point, then we obtain the definition of the dimension $\operatorname{ind} X$ (the space X in this case may be assumed completely regular).

It turns out that:

(a) Always $\operatorname{ind} X \leq \operatorname{ind} X$, $\operatorname{Ind} X \leq \operatorname{Ind} X$, $\operatorname{ind} X \leq \operatorname{Ind} X$.

(b) If in the space X the finite sum theorem holds for the dimension Ind, respectively ind, then $\operatorname{Ind} X = \operatorname{Ind} X$, respectively $\operatorname{ind} X = \operatorname{ind} X$.

(c) For any set $A \subseteq X$ we have $\operatorname{ind} A \leq \operatorname{ind} X$; for any set F closed in X we have $\operatorname{Ind} F \leq \operatorname{Ind} X$.

Theorem 3. If the product $X \times Y$ is paracompact and is an F-product, then

$$\operatorname{Ind} X \times Y \leq \operatorname{Ind} X + \operatorname{Ind} Y.$$

Corollary 4. For almost metrizable spaces (in particular, for paracompacts complete in the sense of Čech, for locally bicompact paracompacts, for bicompacts) X_i, $i = 1, \dots, s$, the formula

$$\operatorname{Ind} X_1 \times \dots \times X_s \leq \operatorname{Ind} X_1 + \dots + \operatorname{Ind} X_s. \tag{***}$$

is valid.

Corollary 5. Formula $(***)$ is valid if all spaces X_i are paracompact and all X_i, except possibly one, are locally bicompact.

Theorem 4. For completely regular spaces one always has

$$\operatorname{ind} X_1 \times \dots \times X_s \leq \operatorname{ind} X_1 + \dots + \operatorname{ind} X_s.$$

II. This part of the note is devoted to the construction of bicompacts with noncoinciding dimensions dim and ind by means of taking the product of bicompacts with coinciding dimensions.

Let a compactum Φ be given. A bicompact $X \supseteq \Phi$ will be called an envelope of the compactum Φ if the following conditions are satisfied:

(1) There exists such a defining system of neighborhoods O_α of the compactum Φ in X, numbered by transfinite numbers α, $1 \leq \alpha_0 \leq \alpha < \omega(c)$, that $O_\alpha \supseteq [O_{\alpha+1}]$ for all α.

(2) X has a retraction r onto Φ.

(3) For arbitrarily large α there exist bicompacts $\Phi_\alpha \subseteq$

$\subseteq O_\alpha \setminus [O_{\alpha+1}]$, that: a) for any point $x \in \Phi$ and any α' there exists a bicompact Φ_α, $\alpha \geq \alpha'$, for which $\dim(\Phi_\alpha \cap r^{-1}x) \geq 1$; b) the mappings $r : \Phi_\alpha \to \Phi$ are open.*

Theorem 5. For a swelling X of a compactum Φ the inequality

$$\operatorname{ind} X \geq \dim \Phi + 1$$

holds.

Theorem 6. For any compactum Φ there exists a swelling X such that:

(a) the retraction $r : X \to \Phi$ is an open mapping;

(b) $\dim X = \max(\dim \Phi, 1)$, and hence $\dim X = \dim \Phi$ for $\dim \Phi \geq 1$;

(c) $\operatorname{ind} X = \operatorname{Ind} X = \dim \Phi + 1$.

Theorem 7. There exists a bicompact Z with $\dim Z = \operatorname{ind} Z = \operatorname{Ind} Z = 2$, for which:

(a) $\dim Z \times Z = 3,\ \operatorname{Ind} Z \times Z \geq \operatorname{ind} Z \times Z \geq 4$;

(b) there exists a compactum B with $\dim B = 2$ such that

$$\dim Z \times B = 3, \qquad \operatorname{Ind} Z \times B \geq \operatorname{ind} Z \times B \geq 4.$$

III. The aim of this part of the note is to introduce the inductive dimensions $\sigma \operatorname{Ind}$ and $\sigma \operatorname{ind}$, which coincide with Ind and ind, respectively, on sufficiently broad classes of spaces, and which outside these classes possess properties better than those of the dimensions ind and Ind.

Definition 2. We put $\sigma \operatorname{Ind} \Lambda = \sigma \operatorname{ind} \Lambda = -1$. Suppose that the class of spaces with $\sigma \operatorname{Ind} X \leq n-1$, respectively with $\sigma \operatorname{ind} X \leq n-1$, has already been defined. We shall say that $\sigma \operatorname{Ind} X \leq n$, respectively $\sigma \operatorname{ind} X \leq n$, if the space X can be represented as a sum of no more than countably many such closed sets X_i, $i = 1, 2, ...$, that for any closed set $F \subseteq X$, respectively any point $x \in X$, and any of its (ee) neighborhoods OF, respectively Ox, for any $i = 1, 2, ...$ for which $F \cap X_i \neq \Lambda$ ($x \in F_i$), there exists such a neighborhood V_i of the set $F \cap X_i$ in X_i, respectively of the point x in X_i, that $V_i \subseteq OF$, respectively $V_i \subseteq Ox$, and $\sigma \operatorname{Ind} \operatorname{Fr} V_i \leq n-1$, respectively $\sigma \operatorname{ind} \operatorname{Fr} V_i \leq n-1$.

1) If the space X is represented as a countable sum of closed-in-X sets X_i with $\sigma \operatorname{Ind} X_i \leq n$, respectively $\sigma \operatorname{ind} X_i \leq n$, $i = 1, 2, ...$, then $\sigma \operatorname{Ind} X \leq n$, respectively $\sigma \operatorname{ind} X_i \leq n$.

2) For a set F closed in X, always $\sigma \operatorname{Ind} F \leq \sigma \operatorname{Ind} X$. For any subset A of a space X, always $\sigma \operatorname{ind} A \leq \sigma \operatorname{ind} X$.

3) Always $\dim X \leq \sigma \operatorname{Ind} X \leq \operatorname{Ind} X$, $\sigma \operatorname{ind} X \leq \operatorname{ind} X$, $\sigma \operatorname{ind} X \leq \sigma \operatorname{Ind} X$. For the Lokuciewski bicompactum L (7): $\dim L \sigma \operatorname{Ind} L = 1 < \operatorname{ind} L \leq \operatorname{Ind} L$. For Roy' s metric space (8) $\sigma \operatorname{ind} R = \operatorname{ind} R = 0 < \dim R = \sigma \operatorname{Ind} R = \operatorname{Ind} R = 1$.

4) If in a space X the countable sum theorem holds for the dimension Ind (ind), then $\sigma \operatorname{Ind} X = \operatorname{Ind} X$ ($\sigma \operatorname{ind} X = \operatorname{ind} X$). Thus, for spaces with a countable base (even for hereditarily finally compact spaces) $\sigma \operatorname{ind} X = \operatorname{ind} X$, and for Dowker (9), in particular metric, spaces $\sigma \operatorname{Ind} X = \operatorname{Ind} X$.

5) For fully paracompact (10) spaces $\dim X \leq \sigma \operatorname{ind} X = \sigma \operatorname{Ind} X \leq \operatorname{ind} X$. Consequently, for strongly metrizable spaces (10) $\dim X = \sigma \operatorname{ind} X = \sigma \operatorname{Ind} X = \operatorname{ind} X = \operatorname{Ind} X$.

6) If a subspace $A \subseteq X$ is fully paracompact, then $\sigma \operatorname{Ind} A \leq \sigma \operatorname{Ind} X$.

We shall call a set $A \subseteq X$ a set of type dF_σ in X if A decomposes into a σ-discrete-in-A system of sets closed in X.

7) If the space X is a σ-discrete sum of closed sets F_α with $\sigma \operatorname{Ind} F_\alpha \leq n$, respectively $\sigma \operatorname{ind} F_\alpha \leq n$, then also $\sigma \operatorname{Ind} X \leq n$, respectively $\sigma \operatorname{ind} X \leq n$.

* This requirement can be weakened. Moreover, swellings can, of course, be constructed not only for compacta, but also for arbitrary bicompacta Φ, taking $1 \leq \alpha_0 \leq \alpha \leq \omega_\tau$ with $\tau = \overline{m}(\Phi)$.

8) If a normal subset $A \subseteq X$ has type dF_σ in X, then $\sigma \operatorname{Ind} A \leq \sigma \operatorname{Ind} X$.

9) If X has a point-finite covering by open sets O_α with $\sigma \operatorname{Ind} O_\alpha \leq n$, then also $\sigma \operatorname{Ind} X \leq n$.

Definition 3. We shall write $\operatorname{loc} \sigma \operatorname{Ind} X \leq n$ if for every point $x \in X$ there exists a neighborhood Ox with $\sigma \operatorname{Ind}[Ox] \leq n$.

10) Always $\operatorname{loc} \sigma \operatorname{Ind} X \leq \sigma \operatorname{Ind} X$. If the space X is weakly paracompact, then $\operatorname{loc} \sigma \operatorname{Ind} X = \sigma \operatorname{Ind} X$.

11) If a closed covering $\{F_\alpha\}$ of the space X is locally countable, then

$$\sigma \operatorname{ind} X \leq \sup_\alpha \sigma \operatorname{ind} F_\alpha,$$

and if, in addition, the space X is weakly paracompact, then

$$\sigma \operatorname{Ind} X \leq \sup_\alpha \sigma \operatorname{Ind} F_\alpha.$$

12) Always

$$\sigma \operatorname{ind} X \times Y \leq \sigma \operatorname{ind} X + \sigma \operatorname{ind} Y.$$

For an F-product $X \times Y$, always

$$\sigma \operatorname{Ind} X \times Y \leq \sigma \operatorname{Ind} X + \sigma \operatorname{Ind} Y.$$

13) For a closed mapping f of a normal space X onto a weakly paracompact space Y, always

$$\dim X \leq \dim f + \sigma \operatorname{Ind} Y.$$

IV. **Theorem 8.** For a paracompactum X, always

$$\operatorname{Ind} X \leq (\dim X + 1) \operatorname{loc} \operatorname{Ind} X \leq (\operatorname{loc} \operatorname{Ind} X + 1) \operatorname{loc} \operatorname{Ind} X.$$

If $\operatorname{loc} \operatorname{Ind} X = 1$, then also $\operatorname{Ind} X = 1$.

Theorem 9. If $\operatorname{Ind} X < \infty$ for a paracompactum X and $\operatorname{Ind} Y < \infty$ for a locally bicompact paracompactum Y, then also $\operatorname{Ind} X \times Y < \infty$. Moreover, for any two numbers $n = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$ there exists a number $k = k(n, m)$ such that, as soon as $\operatorname{Ind} X \leq n$ for a paracompactum X and $\operatorname{Ind} Y \leq m$ for a locally bicompact paracompactum Y, then

$$\operatorname{Ind} X \times Y \leq k(n, m).$$

Theorem 10. For any $n = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$ there exists a number $k = k(n, m)$ such that

$$\operatorname{ind} X \times Y \leq k(\operatorname{ind} X, \operatorname{ind} Y) \leq k(\operatorname{ind} X - 1, \operatorname{ind} Y) + k(\operatorname{ind} X, \operatorname{ind} Y - 1) + 2,$$

with $k(0, n) = k(n, 0) = n$ (X and Y are regular)*.

Note added in proof. V. Filippov has shown that

$$\operatorname{ind} X \times I \leq \operatorname{ind} X + 1$$

for a bicompactum X and an interval I. In connection with this result one can show that

$$\operatorname{ind} X \times Y \leq \operatorname{ind} X + n$$

for an arbitrary space X and a space Y that can be mapped onto a metrizable space R with $\dim R \leq n$ by a decomposing mapping. If the space Y itself is metrizable, then

$$\operatorname{ind} X \times Y \leq \operatorname{ind} X.$$

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* Theorem 10 was obtained by me jointly with A. Mironov.

Note: Figure translations are in progress. See original paper for figures.

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