ON THE INTEGRATION OF THE EQUATIONS OF EQUILIBRIUM OF A CYLINDRICAL SHELL

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Abstract

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ON THE INTEGRATION OF THE EQUATIONS OF EQUILIBRIUM OF A CYLINDRICAL SHELL

The investigation of the elastic equilibrium of a thin or hollow circular cylindrical shell of constant thickness $2h$ and radius R, following the general theory developed in (^{1}), leads to an elliptic system of equations of the 12th order with constant coefficients, which can be written in the following form:

$$\begin{gathered}
\mu\Delta u_1 + (\lambda+\mu)\partial\theta_1/\partial x + \lambda\partial v/\partial x - \varepsilon\lambda\partial u/\partial x = X_1,\\
\mu\Delta u_2 + (\lambda+\mu)\partial\theta_1/\partial y + \lambda\partial v/\partial y - \varepsilon((\lambda+3\mu)\partial u/\partial y + \mu v_2) - \varepsilon^2\mu u_2 = X_2,\\
\mu\Delta v - 12(\lambda+2\mu)v + \lambda\theta_1 + \varepsilon(\lambda\partial v_1/\partial x +\\
+(\lambda+3\mu)\partial v_2/\partial y + 12\lambda u) - \varepsilon^2(\lambda+2\mu)v = X_3,\\
\mu\Delta v_1 + (\lambda+\mu)\partial\theta_2/\partial x - 12\mu(\partial u/\partial x + v_1) - \varepsilon\lambda\partial v/\partial x = X_4,\\
\mu\Delta v_2 + (\lambda+\mu)\partial\theta_2/\partial y - 12\mu(\partial u/\partial y + v_2) -\\
-\varepsilon((\lambda+3\mu)\partial v/\partial y + 12\mu u_2) - \varepsilon^2\mu v_2 = X_5,\\
\mu\Delta u + \mu\theta_2 + \varepsilon(\lambda\partial u_1/\partial x + (\lambda+3\mu)\partial u_2/\partial y + \lambda v) - \varepsilon^2(\lambda+2\mu)u = X_6,\\
(\theta_1 = \partial u_1/\partial x + \partial u_2/\partial y, \quad \theta_2 = \partial v_1/\partial x + \partial v_2/\partial y, \quad \varepsilon = 2h/R).
\end{gathered} \tag{1}$$

Here X_i $(i = 1, \dots, 6)$ are prescribed functions, which are expressed in terms of external surface and body forces; ε is a dimensionless small parameter. The variables x and y are dimensionless isometric coordinates on the cylindrical surface: $ds^2 = 4h^2(dx^2 + dy^2)$: $\xi = 2hx$ is the length of a generator, $\eta = 2hy$ is the length along the circumference of a cylinder of radius R. Thus, by a similarity transformation with center at the point $\xi = \eta = 0$ and modulus $1/2h$, the development of a cylinder of radius R is mapped topologically onto a domain D of the plane of Cartesian coordinates x and y. A plate of constant thickness equal to 1, whose middle plane is the domain D, will be called the model for the cylindrical shell under consideration.

Fig. 1. Distribution of forces and moments on the area $x = \text{const}$

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Figure 1: Fig. 1. Distribution of forces and moments on the area $x = \text{const}$

We shall now explain the geometric meaning of the functions entering into (1): u_1+tv_1, u_2+tv_2 are tangential displacements, and $u+tv$ is the normal deflection of the cylindrical surface of radius $R+2ht$, $-\frac{1}{2} \leq t \leq \frac{1}{2}$. The function $v(x,y)$ determines the elongation (if $v > 0$) or shortening (if $v < 0$) of the transverse fiber passing through the point (x,y) of the middle surface. In classical theory this quantity is neglected (the Kirchhoff-Love hypothesis).

Let T_1, T_2 be the normal stresses; S the shear stress; N_1, N_2 the transverse forces; M_1, M_2 the bending moments; and H the twisting moment, calculated per unit length and applied to the areas $x = \text{const}$, $y = \text{const}$, respectively. In addition to these quantities known from classical theory, we also consider the forces of equilibrated transverse pairs Q_1, Q_2 and the transverse force T. Thus, the physical

the hypothesis on which our considerations are based differs from the classical one. For example, on an elementary area of the cross section $x = \text{const}$, along with the normal and tangential forces T_1 and S, the transverse force N_1, the bending and twisting moments M_1 and H, there also acts an equilibrated transverse pair $(Q_1, -Q_1)$, whose forces lie on the normal to the middle surface and are applied at symmetrically located points (see Fig. 1). For the above-mentioned physical quantities we have the following expressions (^{1,2}):

$$\begin{gathered}
T_1 = (\lambda+2\mu)\,\partial u_1/\partial x + \lambda\,\partial u_2/\partial y + \lambda v - \varepsilon\lambda u,\\
T_2 = \lambda\,\partial u_1/\partial x + (\lambda+2\mu)\,\partial u_2/\partial y + \lambda v - \varepsilon(\lambda+2\mu)u,\\
S = \mu(\partial u_1/\partial y + \partial u_2/\partial x), \qquad N_1 = \mu(\partial u/\partial x + v_1),\\
N_2 = \mu(\partial u/\partial y + v_2 + \varepsilon u_2), \qquad T = \lambda\theta_1 + (\lambda+2\mu)v - \varepsilon\lambda u,\\
M_1 = \frac{1}{6}h((\lambda+2\mu)\,\partial v_1/\partial x + \lambda\,\partial v_2/\partial y - \varepsilon\lambda v),\\
M_2 = \frac{1}{6}h(\lambda\,\partial v_1/\partial x + (\lambda+2\mu)\,\partial v_2/\partial y - \varepsilon(\lambda+2\mu)v),\\
H = -\frac{1}{6}h\mu(\partial v_1/\partial y + \partial v_2/\partial x), \qquad Q_1 = \frac{1}{8}\mu\,\partial v/\partial x,\\
Q_2 = \frac{1}{8}\mu(\partial v/\partial y + \varepsilon v_2).
\end{gathered} \tag{2}$$

If in (1) and (2) we put $\varepsilon = 0$, then we obtain systems of equations of elastic equilibrium of the model plate (^{4}).

The formulas expressing the solution of system (1) for $\varepsilon = 0$, $X_i = 0$ ($i = 1, \dots, 6$), can be written in the form (precisely this form of notation will be used below) ($^{1-3}$)

$$
\begin{aligned}
v &= \Delta\Delta\left(\chi - \frac{\lambda}{\lambda+2\mu}\Delta\varphi\right), \\
u_1 &= \Delta\Delta\left(-\frac{\lambda}{48(\lambda+\mu)}\frac{\partial\chi}{\partial x} + \frac{\partial\varphi}{\partial x} - \frac{\partial\varphi^*}{\partial y}\right), \\
u_2 &= \Delta\Delta\left(-\frac{\lambda}{48(\lambda+\mu)}\frac{\partial\chi}{\partial y} + \frac{\partial\varphi}{\partial y} + \frac{\partial\varphi^*}{\partial x}\right);
\end{aligned} \tag{3}
$$

$$
u = \Delta\Delta(-\omega + B_0\Delta\omega) \qquad (B_0 = (\lambda+2\mu)/12\mu), \tag{4}
$$

$$
v_1 = \Delta\Delta(\partial\omega/\partial x - \partial\psi/\partial y), \qquad v_2 = \Delta\Delta(\partial\omega/\partial y + \partial\psi/\partial x),
$$

where χ and ψ are arbitrary real solutions of the equations

$$
\Delta\chi - k^2\chi = 0, \qquad \Delta\psi - m^2\psi = 0 \quad (k^2 = 48(\lambda+\mu)/(\lambda+2\mu),\ m^2 = 12); \tag{5}
$$

φ, φ^*, and ω are functions satisfying the equation $\Delta^4 w = 0$, but having the following form:

$$
\varphi = \frac{\lambda+2\mu}{64(3\lambda+2\mu)}\left(\frac{z^3}{3}\bar{f} + \frac{\bar{z}^3}{3}f - \frac{z^2}{2}\bar{f}_0 - \frac{\bar{z}^2}{2}f_0\right); \tag{6}
$$

$$
\varphi^* = i\,\frac{2(\lambda+\mu)}{48(3\lambda+2\mu)}\left(z^3\bar{f} - \bar{z}^3 f\right); \tag{7}
$$

$$
\omega = z^3\bar{g} + \bar{z}^3 g + z^2\bar{g}_0 + \bar{z}^2 g_0, \tag{8}
$$

where f, f_0, g, g_0 are arbitrary analytic functions of the complex argument $z = x + iy$.

The presence in (1) and (2) of the small parameter ε regulates the degree of deviation of the stress-state pattern of the circular cylindrical shell from the corresponding state of the model plate. One may expect that, for identical elastic characteristics and external loads, the discrepancy between them will be a quantity of order ε.

Despite the comparatively simple structure of system (1), formulas expressing its general solution in explicit form have not yet been found. Therefore it is necessary to resort to approximate methods of integration. For shallow or thin shells it is expedient to apply the method of expanding the required solutions in a power series with respect to the small parameter

ε. For the coefficients of these expansions we obtain a sequence of systems of equations for the model plate with right-hand sides of a special structure. These systems can be integrated successively in explicit form (^{4}). In this way one can construct approximate solutions of system (1) in the form of polynomials of arbitrary degree with respect to the parameter ε. An analogous device was applied earlier to the case of a spherical shell (^{3}).

To illustrate the method, we shall restrict ourselves to considering approximate solutions that depend linearly on ε. This means that in all calculations we shall neglect terms containing ε to powers higher than the first. It is not superfluous to note that such solutions, taking into account the smallness of the parameter ε, are quite sufficient for many practical applications.

For simplicity we shall assume that the shell is free from external surface and body loads. As is known, the general problem can always be reduced to this special case.

Seeking solutions of system (1) in the form

$$\begin{aligned} v &= v^0 + \varepsilon v', \qquad & u_1 &= u_1^0 + \varepsilon u_1', \qquad & u_2 &= u_2^0 + \varepsilon u_2', \\ u &= u^0 + \varepsilon u', \qquad & v_1 &= v_1^0 + \varepsilon v_1', \qquad & v_2 &= v_2^0 + \varepsilon v_2', \end{aligned} \tag{9}$$

we obtain that $v^0, u_1^0, u_2^0, u^0, v_1^0, v_2^0$ satisfies system (1) for $\varepsilon = 0$ and $X_i = 0$ $(i = 1, \ldots, 6)$, while $v', u_1', u_2', u', v_1', v_2'$ is a solution of the same system for $\varepsilon = 0$, but with right-hand sides of the form

$$\begin{aligned} &X_1 = \lambda\, \partial u^0/\partial x, \qquad X_2 = (\lambda + 3\mu)\, \partial u^0/\partial y + \mu u_2^0, \\ &X_3 = -\lambda\, \partial v_1^0/\partial x - (\lambda + 3\mu)\, \partial v_2^0/\partial y - 12\lambda u^0, \\ &X_4 = \lambda\, \partial v^0/\partial x, \qquad X_5 = (\lambda + 3\mu)\, \partial v^0/\partial y + 12\mu u_2^0, \\ &X_6 = -\lambda\, \partial u_1^0/\partial x - (\lambda + 3\mu)\, \partial u_2^0/\partial y - \lambda v^0. \end{aligned} \tag{10}$$

If we now represent the functions $v^0, u_1^0, u_2^0, u^0, v_1^0, v_2^0$ by formulas (3) and (4), then we shall have

$$X_1 = \Delta\Delta \left(-\lambda \frac{\partial \omega}{\partial x} + \lambda B_0 \frac{\partial \Delta\omega}{\partial x} \right),$$

$$X_2 = \Delta\Delta \left(\mu \frac{\partial \psi}{\partial x} - (\lambda + 2\mu) \frac{\partial \omega}{\partial y} + (\lambda + 3\mu) B_0 \frac{\partial \Delta\omega}{\partial y} \right),$$

$$X_3 = \Delta\Delta \left(-3\mu \frac{\partial^2 \psi}{\partial x \partial y} - (\lambda + 12\lambda B_0)\Delta\omega - 3\mu \frac{\partial^2 \omega}{\partial y^2} + 12\lambda\omega \right), \tag{11}$$

$$X_4 = \Delta\Delta \left(\lambda \frac{\partial \chi}{\partial x} - \frac{\lambda^2}{\lambda + 2\mu} \frac{\partial \Delta\varphi}{\partial x} \right),$$

$$X_5 = \Delta\Delta\left(\left(\lambda + 3\mu - \frac{\mu\lambda}{4(\lambda+\mu)}\right)\frac{\partial\chi}{\partial y} + 12\mu\frac{\partial\varphi}{\partial y} - \frac{\lambda(\lambda+3\mu)}{\lambda+2\mu}\frac{\partial\Delta\varphi}{\partial y} + 12\mu\frac{\partial\varphi^*}{\partial x}\right),$$

$$X_6 = \Delta\Delta\left(\frac{\mu\lambda}{48(\lambda+\mu)}\left(-2\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2}\right) - \frac{2\mu\lambda}{\lambda+2\mu}\frac{\partial^2\varphi}{\partial x^2} - \frac{\mu(5\lambda+6\mu)}{\lambda+2\mu}\frac{\partial^2\varphi}{\partial y^2} - 3\mu\frac{\partial^2\varphi^*}{\partial x\partial y}\right).$$

By means of rather lengthy, but essentially simple calculations, one can verify (see (^{3})) that system (1) for $\varepsilon = 0$ and with right-hand sides of the form (11) admits a particular solution of the form

$$\hat v = \frac{24(\lambda+3\mu)}{3\lambda+2\mu}\frac{\partial^2\psi}{\partial x\partial y} + \Delta L_{kP}\omega; \tag{12}$$

$$\hat u_1 + i\hat u_2 = 2\frac{\partial}{\partial\bar z}\left\{\frac{\mu-2\lambda}{3\lambda+2\mu}\frac{\partial^2\psi}{\partial x\partial y} - \frac{1}{\lambda+2\mu}(\lambda L_{kP} - \mu Q)\omega + i\left(\frac{\partial^2\psi}{\partial x^2} - (2-3B_0\Delta)\frac{\partial^2\omega}{\partial x\partial y}\right)\right\}; \tag{13}$$

$$\begin{aligned}\hat u = &-\frac{\lambda(6\lambda+5\mu)}{2(\lambda+\mu)(\lambda+2\mu)}\frac{\partial^2\chi}{\partial x^2} - \frac{3\mu}{\lambda+2\mu}\frac{\partial^2\chi}{\partial y^2} - \frac{\mu}{\lambda+2\mu}\left(S\varphi - 24\frac{\partial^2\varphi^*}{\partial x\partial y}\right)\\ &-\frac{2\lambda}{\lambda+2\mu}\frac{\partial^2\Delta\varphi}{\partial x^2} - \frac{5\lambda+6\mu}{\lambda+2\mu}\frac{\partial^2\Delta\varphi}{\partial y^2} - 3\frac{\partial^2\Delta\varphi^*}{\partial x\partial y};\end{aligned} \tag{14}$$

$$\begin{aligned}\hat v_1 + i\hat v_2 = 2\frac{\partial}{\partial z}\Biggl\{&\frac{\lambda(2\lambda+\mu)}{2(\lambda+\mu)(\lambda+2\mu)}\frac{\partial^2\chi}{\partial x^2} + \frac{\lambda+3\mu}{\lambda+2\mu}\frac{\partial^2\chi}{\partial y^2} +\\ &+\frac{\mu}{\lambda+2\mu}\left(S\varphi - 24\frac{\partial^2\varphi^*}{\partial x\,\partial y}\right) + i\Biggl(\frac{11\lambda+12\mu}{3\lambda+2\mu}\frac{\partial^2\chi}{\partial x\,\partial y} +\\ &+L_m\left(\left(12\Delta - \frac{3\lambda}{\lambda+2\mu}\right)\Delta\Delta\right)\frac{\partial^2\varphi}{\partial x\,\partial y} + 12\frac{\partial^2\Delta\varphi^*}{\partial x^2}\Biggr)\Biggr\},\end{aligned} \tag{15}$$

where L_k denotes a particular solution of the equation $\Delta w - k^2 w = g$ (see (4));

$$\begin{aligned}P &= \left(-\frac{24\lambda}{\lambda+2\mu} + \frac{3(\lambda-\mu)}{\mu}\right)\frac{\partial^2}{\partial y^2} + \left(\frac{24\lambda}{\lambda+2\mu}\Delta + \frac{3\lambda}{\mu}\Delta\Delta\right),\\ Q &= (-2+3B_0\Delta)\frac{\partial^2}{\partial y^2} - \frac{\lambda}{\mu}(\Delta - B_0\Delta\Delta);\end{aligned} \tag{16}$$

$$S = -\frac{24\lambda}{\lambda+2\mu}\frac{\partial^2}{\partial x^2} - \frac{48(\lambda+\mu)}{\lambda+2\mu}\frac{\partial^2}{\partial y^2} - \frac{\lambda^2}{\mu(\lambda+2\mu)}\frac{\partial^2\Delta}{\partial x^2} - \frac{\lambda(\lambda+3\mu)}{\mu(\lambda+2\mu)}\frac{\partial^2\Delta}{\partial y^2}.$$

$$(2\partial/\partial\bar{z} = \partial/\partial x + i\partial/\partial y, \quad 2\partial/\partial z = \partial/\partial x - i\partial/\partial y).$$

It is now not difficult to see that the sought functions (9) have the form

$$\begin{gathered} v = v^0 + \varepsilon(\tilde{v} + \hat{v}), \quad u_1 = u_1^0 + \varepsilon(\tilde{u}_1 + \hat{u}_1), \quad u_2 = u_2^0 + \varepsilon(\tilde{u}_2 + \hat{u}_2), \qquad (17) \\ w = w^0 + \varepsilon(\tilde{u} + \hat{u}), \quad v_1 = v_1^0 + \varepsilon(\tilde{v}_1 + \hat{v}_1), \quad v_2 = v_2^0 + \varepsilon(\tilde{v}_2 + \hat{v}_2), \end{gathered}$$

where $\tilde{v}, \tilde{u}_1, \tilde{u}_2, \tilde{u}, \tilde{v}_1, \tilde{v}_2$ are an arbitrary solution of the homogeneous system of equations for the model plate.

Approximate solutions of the form (17) can be used to solve various boundary-value problems. Formulas (17) make it possible to reduce the solution of boundary-value problems for cylindrical shells to two one-type boundary-value problems of elastic equilibrium of a model plate.

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Note: Figure translations are in progress. See original paper for figures.

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