SOME TOPOLOGICAL QUESTIONS FOR FINITE-DIMENSIONAL VECTOR FIELDS RELATED TO THE THEORY OF DISCONTINUOUS SOLUTIONS OF SYSTEMS OF QUASILINEAR CONSERVATION LAWS

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Abstract

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E. B. BYKHOVSKII

SOME TOPOLOGICAL QUESTIONS FOR FINITE-DIMENSIONAL VECTOR FIELDS RELATED TO THE THEORY OF DISCONTINUOUS SOLUTIONS OF SYSTEMS OF QUASILINEAR CONSERVATION LAWS

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In the theory of discontinuous solutions of systems of quasilinear conservation laws

$$\frac{\partial L(u)}{\partial t} + \sum_{r=1}^{n} \frac{\partial F^r(u)}{\partial x_r} = 0, \tag{1}$$

where

$$L(u),\ F^r(u) \qquad (r = 1, 2, \dots, n) \tag{2}$$

are N-dimensional vector fields in the space $u = (u_1, \dots, u_N)$, the following problem arises. If $\nu = (\nu_t, \nu_{x_1}, \dots, \nu_{x_n})$ is the normal to the discontinuity of the solution, $\omega = (\omega_1, \dots, \omega_n)$; $\omega_r = \nu_{x_r}/\nu_t$, and u^+, u^- are respectively the upper and lower limiting values of u at the discontinuity, then (1) at the point of discontinuity can be written in the form

$$L(u^+) - L(u^-) + \sum_r \omega_r \left[F^r(u^+) - F^r(u^-)\right] = 0.$$

Fixing u^- and regarding u^+ as variable, we have

$$L(u) - L(u^-) + \sum_r \omega_r \left[F^r(u) - F^r(u^-)\right] = 0. \tag{3}$$

For every value of the parameter $\vec{\omega}$, equality (3) has the trivial solution $u = u^-$. Of interest is the study of the nontrivial solution $u(\vec{\omega})$ (the multivaluedness of this function is not excluded), which is an $\vec{\omega}$-parametrized manifold in the space u.

In the simplest case $r = 1$, this manifold is customarily called the Hugoniot curve. In the general case we shall call it the manifold G. We shall call the system of fields (2) and the system (1) potential if these fields are potential: $L(u) = \mathcal{L}'(u)$; $F^r(u) = \mathcal{F}^{r\prime}(u)$, where $\mathcal{L}(u)$, $\mathcal{F}^r(u)$ are scalar functions, and the prime denotes taking the gradient. We shall call the systems (2) and (1) reducible if there exists (in the region of the space u under consideration) a nondegenerate transformation $u = u(q)$ such that $L(u(q)), F^r(u(q))$ is a potential system. A number of authors have noted that systems frequently encountered in physics are potential or reducible. Thus, S. K. Godunov ([1]) indicated that the system of conservation laws of mass, momentum, and energy in gas dynamics is reducible.* In ([1]) it is noted that if (1) is a gas-dynamical system reduced to potential form, then scalar multiplication of it by u leads, for smooth solutions, to the entropy conservation law

$$\frac{\partial}{\partial t}\left[(u, \mathcal{L}'(u)) - \mathcal{L}(u)\right] + \sum_r \frac{\partial}{\partial x_r}\left[(u, \mathcal{F}^{r\prime}(u)) - \mathcal{F}^r(u)\right] = 0.$$

* It is curious to note that the method of reduction given by Godunov ceases to be applicable under simplifying assumptions on the equation of state (of the barotropic type), for in that case $u(q)$ degenerates identically.

It is natural to ask whether the corresponding inequality

$$\frac{\partial}{\partial t}\left[(u, \mathcal{L}'(u)) - \mathcal{L}(u)\right] + \sum_r \frac{\partial}{\partial x_r}\left[(u, \mathcal{F}^{r\prime}(u)) - \mathcal{F}^r(u)\right] \le 0 \qquad (4)$$

(the derivatives for discontinuous solutions are understood as measures; see (2)) is thereby an analogue of the entropy-increase condition, which in the general case ensures uniqueness of discontinuous solutions. The answer to this question is essentially connected with the topological structure of the G-manifold, as the following example shows. Let system (1) be potential,

$$r = 1; \qquad \mathcal{L}(u) = \frac{1}{2}\sum_{i=1}^{N} u_i^2, \qquad \mathcal{F}(u) = \sum_{i=1}^{N} \int \mathcal{F}_i(u_i)\, du_i,$$

where $\mathcal{F}_i(u_i)$ are smooth functions of their arguments, convex downward. Then (1) decomposes into N separate well-studied equations

$$\partial u_i/\partial t + \partial \mathcal{F}_i(u_i)/\partial x = 0, \qquad (5)$$

for each of which the known uniqueness condition is

$$\omega(u_i^+ - u_i^-) \leq 0. \tag{6}$$

It is not difficult to see that (6), under our convexity assumptions, is equivalent to condition (4), written at a discontinuity for each equation separately:

$$A_i \stackrel{d}{=} \frac{1}{2}[(u_i^+)^2 - (u_i^-)^2] - \omega \left[\int_{u_i^-}^{u_i^+} \mathcal{F}_i(\xi)\, d\xi - u_i^+ \mathcal{F}_i(u_i^+) + u_i^- \mathcal{F}_i(u_i^-) \right] \leq 0. \tag{7}$$

On the other hand, if (5) is regarded as a system of equations, then (4) for it can be written in the form

$$\sum_{i=1}^{N} A_i \leq 0, \tag{8}$$

which, generally speaking, is less restrictive than the N conditions

$$A_i \leq 0 \qquad (i = 1, 2, \dots, N).$$

Nevertheless, (8) still ensures uniqueness if the curve G is sufficiently regular, but may fail to ensure it under a certain "pathology" in the structure of G. Indeed, (3) has the form of the equalities

$$u_i - u_i^- + \omega[\mathcal{F}_i(u_i) - \mathcal{F}_i(u_i^-)] = 0 \qquad (i = 1, 2, \dots, N). \tag{9}$$

They can be satisfied by setting $u_i \equiv u_i^-$ for $i = 1, 2, \dots, N;\ i \neq k$, and taking $u_k \neq u_k^-$, with u_k satisfying (9). Letting successively $k = 1, 2, \dots, N$, we obtain N branches of the curve G, which are straight lines passing through the point u^- parallel to the coordinate axes; moreover, on the branch parallel to the u_k-axis, the point u^- is obtained for $-1/\omega = \mathcal{F}'_k(u_k^-)$. We shall call these branches principal.

It is possible, however, to take $u_i \equiv u_i^-$ for all i except two, except three, etc., and for these latter write (9) and regard them as not identically equal to u_k^-. Additional branches are obtained, generally speaking not passing through u^-, but for some choice of the point u^- passing through it. Already in the case $N = 2$ it is clear that the $\mathcal{F}_i(u_i)$ can be chosen so that condition (8) does not ensure uniqueness of the solution, because of the possibility of a jump from u^- to an additional branch. If, however, the $\mathcal{F}_i(u_i)$ are chosen so that there are no additional branches (this is possible), then (9) ensures uniqueness of the solution, since at each jump only one component of u is discontinuous, so that in (8) exactly one term is always nonzero.

Next the topological structure of the manifold G is studied in a neighborhood of the point u^-. In doing so, the basic methods of bifurcation theory set forth in (3) prove useful.

Notation and terminology: $L'(u)$ is the matrix $\partial L_i/\partial u_k$; $L''(u)$ is the tensor of second derivatives of $L_i(u)$. Analogous notation is introduced for $F^r(u)$. Contraction over one of the indices (in each case it is clear which one) is denoted by ordinary multiplication, so that if v is an N-vector, then $L'(u)v$ is an N-vector, $L''(u)v$ is an $N \times N$ matrix, and $L''(u)vv$ is the N-vector obtained by applying this matrix to v. The scalar product of N-vectors is denoted by (,). $\vec{\omega} = (\omega_1, \dots, \omega_n)$; $\vec{\mu} = (\mu_1, \dots, \mu_n)$ are n-dimensional vectors. The vector μ is an eigenvalue of the linear operator $L'(u) + \sum_r \omega_r F^{r'}(u)$ if, for $\vec{\omega} = \vec{\mu}$, the equation

$$\left[L'(u) + \sum_r \omega_r F^{r'}(u)\right] v = 0$$

has a nontrivial solution v, an eigenvector, which we shall always take to be normalized. By the multiplicity of an eigenvalue we shall mean the number of linearly independent eigenvectors corresponding to it.*

Let μ and v be an eigenvalue and an eigenvector of the operator $L'(u) + \omega F'(u)$. They coincide with those for the operator $E + \omega[L'(u)]^{-1}F'(u)$. Further, let E_0 be the corresponding invariant subspace of the operator $[L'(u)]^{-1}F'(u)$; let E^0 be the complementary invariant subspace; and let P_0 and P^0 be the projectors onto them. In the self-adjoint case $[L'(u)]^{-1}F'(u)$, the projection is orthogonal. With respect to the field $L(u)$ it is assumed that, in the region of the space u under consideration, $\det L'(u) \neq 0$, and that the sum of the multiplicities of the negative eigenvalues of $L'(u)$ does not depend on u. We first consider the case $n = 1$.

Theorem 1. Let, in some neighborhood of the point u^-, the operator $L'(u) + \mu F'(u)$ have a simple real eigenvalue $\mu_0(u)$, with corresponding eigenvector $v(u)$, smoothly depending on u, and suppose that the following conditions are satisfied:

1. (Lax (4)) $(\operatorname{grad} \mu_0(u^-), v_0) \neq 0$, where $v_0 = v(u^-)$.
2. $(F'(u^-)v_0, w_0) \neq 0$, where $w(u)$ is the eigenvector of the operator $[L'(u) + \mu_0 F'(u)]^*$; $w_0 = w(u^-)$.
3. u^- is an isolated zero of the vector field

$$L(u) - L(u^-) + \mu_0(u^-)[F(u) - F(u^-)].$$

Then, for $\omega_0 = \mu_0(u^-)$, the field $L(u) - L(u^-) + \omega[F(u) - F(u^-)]$ has a bifurcation, realized in one of the following three ways:

a) There exists a unique bifurcation branch $u(\omega)$, defined and continuous on some interval $[\omega_0 - \varepsilon, \omega_0 + \varepsilon]$, equal to u^- only for $\omega = \omega_0$, and smooth for all ω, except possibly $\omega = \omega_0$. The curve $u(\omega)$ at $\omega = \omega_0$ is tangent to the vector v_0.

b) There exist exactly two functions $u_1(\omega)$ and $u_2(\omega)$, defined on $[\omega_0, \omega_0 + \varepsilon]$, equal to u^- only for $\omega = \omega_0$, and possessing the properties indicated in a), while for $\omega < \omega_0$ close to ω_0 there are no bifurcation branches.

c) The situation is analogous to b), but $u_1(\omega)$ and $u_2(\omega)$ exist on the interval $[\omega_0 - \varepsilon, \omega_0]$.

Theorem 2. Let the fields $L(u)$, $F(u)$ be potential, $L'(u)$ be definite, and let the preamble and condition 1 of Theorem 1 be satisfied; or else let $L(u)$, $F(u)$ be arbitrary, but let conditions 1 and 2 of Theorem 1 be satisfied, and let condition 3 be replaced by the following:

$3'$. $P_0[L''(u^-) + \omega_0 F''(u^-)]v_0 v_0 \neq 0$, where P_0 is the projector onto the null eigenspace of the operator $L'(u^-) + \omega_0 F'(u^-)$ (one-dimensional, with basis v_0).

Then assertion a) of Theorem 1 holds.

* For our purposes such a definition is sufficient. It is known that, in the general case, multiplicity is defined differently and may be greater than the number of eigenvectors.

From Theorems 1 and 2 it follows

Theorem 3. *Suppose that for the fields $L(u)$, $F(u)$ in a neighborhood of u^- the following conditions are satisfied:*

Either the fields are potential, all N characteristic numbers μ_k of the operator $L'(u^-) + \mu F'(u^-)$ are distinct, $L'(u)$ is sign-definite, and

$$(\operatorname{grad} \mu_k(u^-), v_k) \neq 0 \quad (v_k \text{ are eigenvectors}). \tag{10}$$

Or, in the general case, in a neighborhood of u^- there exist N real distinct characteristic numbers $\mu_k(u)$ of the operator $L'(u) + \mu F'(u)$, (10) is satisfied, and at the point u^- condition 2 of Theorem 1 and condition $3'$ of Theorem 2 are also satisfied.

Then the manifold G in a neighborhood of u^- consists of N branches tangent to the corresponding eigenvectors, and such that, for $\omega = \mu_k(u^-)$, they pass through u^-.

We now consider the general case $n > 1$. For the operator

$$L'(u^-) + \sum_r \omega_r F^{r\prime}(u^-)$$

we shall assume that its characteristic numbers form, in the space $\vec{\omega}$, N isolated spectral surfaces S_k $(k = 1, 2, \dots, N)$, and that each characteristic number is simple. This is equivalent to the existence, for system (1) at the point u^-, of N characteristic conoids. Let S be a spectral surface; $\vec{\omega}^0$ a point on it; v_0 the corresponding eigenvector; w_0 an eigenvector of the adjoint operator; P_0 the projector onto v_0 of the same kind as in Theorem 2. It is not difficult, in some way, to construct in a neighborhood of the point u^- the characteristic number $\mu(\vec{\omega})(u)$ "of the same number" as $\vec{\omega}^0$, $\mu(u^-) = \omega^0$, the corresponding field of eigenvectors $v(u)$, and its vector line passing through u^-. The eigenvectors corresponding to the points of S from a neighborhood of $\vec{\omega}^0$ form a bundle depending on $n-1$ parameters.

Theorem 4. *Suppose:*

1. *Under an infinitesimal displacement along the vector line $v(u)$ indicated above, the change $|\Delta\vec{\mu}|$ has the same order of smallness as the displacement. This is an analogue of condition 1 of Theorem 1.*
2. *The vector* $\mathbf{a}$*:* $a_r = (F^{r'}(u^-)v_0, w_0)$ *is not tangent to S at the point $\vec{\omega}_0$. This is an analogue of condition 2 of Theorem 1.*
3. $$P_0 \left[L''(u') + \sum_r \omega_r^0 F^{r''}(u^-) \right] v_0 v_0 \neq 0.$$

Then the function $u(\vec{\omega})$ defining the G-manifold maps an n-dimensional neighborhood of the point $\vec{\omega}^0$ in the space u in the following way. Each segment of the normal to S intersecting S at a point $\vec{\omega}$ sufficiently close to $\vec{\omega}^0$ is mapped into a curve (a bifurcation branch) passing through u^- and tangent at the point u^- to the eigenvector corresponding to the characteristic number $\vec{\omega}$. At the same time all points of S from a neighborhood of $\vec{\omega}^0$ are mapped to the point u^-.

Thus, each spectral surface S_k generates its own branch of the manifold $u(\omega)$, just as, for $n = 1$, each characteristic number generated its own branch of the curve G.

The proof of all these theorems uses the methods of bifurcation theory set forth in (3), with allowance for the indices of the bifurcation branches. An additional point consists in using the finite-dimensionality of the space u in studying the behavior of these branches.

Leningrad State University
named after A. A. Zhdanov

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Note: Figure translations are in progress. See original paper for figures.