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**Abstract**

**Full Text**

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## CYBERNETICS AND CONTROL THEORY

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### FINITE CONTROL OF DISCRETE LINEAR SYSTEMS

*(Presented by Academician B. N. Petrov, February 21, 1969)*

In <sup>(1-4)</sup>, problems of finite control of linear systems with lumped and distributed parameters were considered. Here we consider the control of discrete linear systems of the form

$$x(n+1) = Ax(n) + Bu(n) + F(n), \quad n = 0, 1, 2, \dots, \quad (1)$$

where  $x$  is a single-column vector (matrix) of the state in the  $k$ -dimensional complex phase space  $X_k$ ;  $u$  is a scalar complex control (vector control is considered analogously);  $A$  is a  $(k \times k)$ -matrix with complex constant elements;  $B$  is a single-column vector (matrix) with complex constant elements;  $F$  is a single-column vector (matrix) with complex elements.

**Problem of finite control.** Let  $N$  be given and

$$x(0) = x_c. \quad (2)$$

Find a lattice function  $u(n)$ ,  $n = 0, 1, 2, \dots, N$ , such that

$$x(N+1) = 0. \quad (3)$$

To solve this problem we shall use the  $Z$ -transform <sup>(5)</sup>. The  $Z$ -transform  $\tilde{y}(z)$  of a function  $y(n)$ ,  $n = 0, 1, 2, \dots$ , is defined by the formula

$$\tilde{y}(z) = Z[y(n)] = \sum_{n=0}^{\infty} y(n)z^n. \quad (4)$$

Apply the  $Z$ -transform to (1), (2). We obtain

$$\frac{1}{z}\tilde{x}(z) - \frac{1}{z}x_0 = A\tilde{x}(z) + B\tilde{u}(z) + \tilde{F}(z). \quad (5)$$

Hence

$$\tilde{x}(z) = \frac{C(1/z)[B\tilde{u}(z) + x_0/z + \tilde{F}(z)]}{|(1/z)E - A|} = \frac{z^k C(1/z)[B\tilde{u}(z) + x_0/z + \tilde{F}(z)]}{|E - Az|}, \quad (6)$$

where  $C(w)$  is the adjugate matrix of the matrix  $A$  (6), and  $E$  is the identity matrix.

In order that  $x(n)$  be a finite function, i.e., have finite support  $n = 0, 1, 2, \dots, N$ , and  $x(n) = 0$  for  $n < 0$  and  $n > N$ , it is necessary and sufficient that the right-hand side of formula (6) be a polynomial in  $z$  of degree  $N$ , or that the polynomial in the numerator of (6) be divisible without remainder by the polynomial of the denominator. The latter condition, in turn, is equivalent to the conditions

$$\left\{ z^k C\left(\frac{1}{z}\right) \left[ B\tilde{u}(z) + \frac{x_0}{z} + \tilde{F}(z) \right] \right\}_{z=z_p}^{(q)} = 0, \quad q = 0, 1, \dots, \alpha_p - 1; \quad p = 1, 2, \dots, m, \quad (7)$$

where  $z_p$  is a root of multiplicity  $\alpha_p$  of the polynomial  $P(z) = |E - Az|$ , and

$$\sum_{p=1}^m \alpha_p = k. \quad (8)$$

The equalities (7) are equivalent to the conditions

$$\tilde{u}^{(q)}(z_p) = \frac{[z^{k-1}C(1/z)x_0 + z^k C(1/z)\tilde{F}(z)]_{z=z_p}^{(q)}}{[z^k C(1/z)B]_{z=z_p}^{(q)}} = \beta_p^q, \quad q = 0, 1, \dots, \alpha_p - 1, \quad p = 1, 2, \dots, m. \quad (9)$$

Thus, in order to determine the desired finite control  $u(n)$ ,  $n = 0, 1, 2, \dots, N$ , it is necessary and sufficient to find a polynomial of degree  $N$  that would be a solution of the interpolation problem (9). Such a polynomial can be constructed using the Lagrange-Hermite formula (7),

$$\tilde{u}(z) = \sum_{p=1}^m \frac{P(z)}{(z - z_p)^{\alpha_p}} \sum_{q=0}^{\alpha_p - 1} \beta_p^q \frac{(z - z_p)^q}{q!} \left[ \frac{(z - z_p)^{\alpha_p}}{P(z)} \right]_{z=z_p}^{(\alpha_p - 1 - q)} + Q(z), \quad (10)$$

where  $Q(z)$  is an arbitrary polynomial of degree  $N$  having the same roots  $z_p$  with the same multiplicities  $\alpha_p$  as the polynomial  $P(z)$ , i.e.,  $Q(z)$  must be divisible by  $P(z)$ .

In the case when the  $z_p$  are simple roots,  $p = 1, 2, \dots, k$ , the formula for the desired  $\tilde{u}(z)$  is simpler:

$$\tilde{u}(z) = \sum_{p=1}^k \beta_p^0 \frac{P(z)}{P'(z_p)(z - z_p)} + Q(z) = \sum_{n=0}^N a_{nz}^n. \quad (11)$$

As a result,  $\tilde{u}(z)$  is represented by a polynomial of degree  $N$ ,

$$\tilde{u}(z) = \sum_{n=0}^N a_{nz}^n. \quad (12)$$

Thus, the desired finite control is given by the formula

$$u(n) = a_n, \quad n = 0, 1, 2, \dots, N. \quad (13)$$

If  $X_k$  is a real space and  $A$ ,  $B$ , and  $F$  are real matrices, then the real finite control is determined by the formula

$$u(n) = \operatorname{Re} a_n, \quad n = 0, 1, 2, \dots, N. \quad (14)$$

Since, for the solution of the interpolation problem (9), it is necessary and sufficient that the quantities  $\beta_p^q$  be finite numbers,  $\beta_p^q \neq \infty$ ,  $q = 0, 1, \dots, \alpha_p - 1$ ,  $p = 1, 2, \dots, m$ , it follows that for controllability of the system (1) it is necessary and sufficient that the conditions

$$\left[ z^{kC} \begin{pmatrix} 1 \\ z \end{pmatrix} B \right]_{z=z_p}^{(q)} \neq 0, \quad q = 0, 1, \dots, \alpha_p - 1; \quad p = 1, 2, \dots, m. \quad (15)$$

be satisfied.

Since the minimal degree of the polynomial solving the interpolation problem (9), generally speaking, for arbitrary  $x_c$ , is equal to  $N$ , it follows that:

- 1) for  $N < k - 1$  the finite-control problem, generally speaking, for arbitrary  $x_c$ , has no solution;
- 2) for  $N = k - 1$  the finite-control problem has a unique solution;
- 3) for  $N > k - 1$ , the finite-control problem has an infinite set of solutions, due to arbitrariness in the choice of the polynomial  $Q(z)$ .

Formula (10) describes the set of the sought finite controls.

Let us note that in the last case, when  $N > k - 1$ , the arbitrariness in the choice of the polynomial  $Q(z)$  can be used to optimize the finite control  $u(n)$ ,  $n = 0, 1, \dots, N$ , according to a prescribed criterion.

To illustrate the method, let us consider the simplest examples.

**Example 1.** The controlled system is described by the equations

$$x(n+1) = x(n) + u(n), \quad n = 0, 1, \dots, N,$$

$$x(0) = x_0, \quad x(N+1) = 0.$$

To determine the finite control  $u(n)$ ,  $n = 0, 1, \dots, N$ , we perform the  $Z$ -transform of the given system. We obtain

$$\frac{1}{z} \tilde{x}(z) - \frac{1}{z} x_0 = \tilde{x}(z) + \tilde{u}(z).$$

Hence

$$\tilde{x}(z) = (z\tilde{u}(z) + x_0)/(1 - z).$$

The root of the polynomial in the denominator of this fraction is

$$z_1 = 1.$$

Therefore the condition

$$\tilde{u}(1) + x_0 = 0$$

must be satisfied.

Hence we obtain the interpolation problem

$$\tilde{u}(1) = -x_0.$$

Thus, the system is controllable.

By formula (11) we obtain that the  $Z$ -transform of the sought finite control  $u(n)$  has the form

$$\tilde{u}(z) = -x_0 + Q(z),$$

where  $Q(z)$  is an arbitrary polynomial of degree  $N$  divisible by  $1-z$ , or, in other words, having the root  $z_1 = 1$ , i.e.  $Q(1) = 0$ .

The last formula can also be written in another form,

$$\tilde{u}(z) = -x_0 + (1-z)S(z),$$

where  $S(z)$  is an arbitrary polynomial of degree  $N-1$ . In particular, for  $N = 0$ ,

$$\tilde{u}(z) = -x_0$$

and, consequently, the finite control is

$$u(0) = -x_0.$$

**Example 2.** A system is given:

$$x_1(n+1) = x_1(n) - x_2(n), \quad x_2(n+1) = x_1(n) + x_2(n) + u(n),$$

$$x_1(0) = x_{01}, \quad x_2(0) = x_{02},$$

$$x_1(N+1) = 0, \quad x_2(N+1) = 0.$$

We perform the  $Z$ -transform of this system. We obtain

$$\frac{1}{z}\tilde{x}_1 - \frac{1}{z}x_{01} = \tilde{x}_1 - \tilde{x}_2,$$

$$\frac{1}{z}\tilde{x}_2 - \frac{1}{z}x_{02} = \tilde{x}_1 + \tilde{x}_2 + \tilde{u}.$$

Solving this linear system with respect to  $\tilde{x}_1$  and  $\tilde{x}_2$ , we obtain

$$\tilde{x}_1 = \frac{z^2\tilde{u} - x_{01}z + x_{01} + x_{02}z}{2z^2 - 2z + 1},$$

$$\tilde{x}_2 = \frac{z^2\tilde{u} - z\tilde{u} + x_{01}z + x_{02}z - x_{02}}{2z^2 - 2z + 1}.$$

The polynomial  $2z^2 - 2z + 1$  has two roots

$$z_1 = \frac{1}{2} + \frac{1}{2}j, \quad z_2 = \frac{1}{2} - \frac{1}{2}j.$$

The numerators in the expressions for  $\tilde{x}_1$  and  $\tilde{x}_2$  must vanish for  $z = z_1$  and  $z = z_2$ . Consequently,

$$z_i^2 \tilde{u}(z_i) - x_{01}z_i + x_{01} + x_{02}z_i = 0, \quad i = 1, 2,$$

$$z_i^2 \tilde{u}(z_i) - z_i \tilde{u}(z_i) + x_{01}z_i + x_{02}z_i - x_{02} = 0, \quad i = 1, 2.$$

From the first condition we obtain

$$\tilde{u}(z_i) = \frac{1 - z_i}{z_i^2} x_{01} - \frac{x_{02}}{z_i}, \quad i = 1, 2.$$

From the second condition we obtain

$$\tilde{u}(z_i) = -\frac{x_{01}}{z_i - 1} - \frac{x_{02}}{z_i}, \quad i = 1, 2.$$

It is easy to see that the right-hand sides of the last two formulas are finite numbers. Consequently, the system is controllable. Moreover, as was to be expected, the right-hand side of the penultimate formula is equal to the right-hand side of the last formula. Such a situation will always occur in the case of a controllable system. Hence there follows an important practical conclusion: after the  $Z$ -transform it is sufficient to solve the algebraic system with respect to only one component  $\tilde{x}_i$ , expressing it in terms of  $x_0$ ,  $\tilde{u}$ , and  $F$ .

Thus, we obtain the following interpolation problem:

$$\tilde{u}\left(\frac{1}{2} + \frac{1}{2}j\right) = -(1 + j)x_{01} - (1 - j)x_{02} = \beta_1,$$

$$\tilde{u}\left(\frac{1}{2} - \frac{1}{2}j\right) = -(1 - j)x_{01} - (1 + j)x_{02} = \beta_2.$$

The interpolation polynomial has the form

$$\tilde{u}(z) = \frac{\beta_1 \left(z - \frac{1}{2} + \frac{1}{2}j\right)}{2j} + \frac{\beta_2 \left(z - \frac{1}{2} - \frac{1}{2}j\right)}{2j} + Q(z),$$

where  $Q(z)$  is an arbitrary polynomial of degree  $N > 1$  (since for the given system  $k = 2$ ), having the roots  $z_1 = \frac{1}{2} + \frac{1}{2}j$  and  $z_2 = \frac{1}{2} - \frac{1}{2}j$ .

Hence

$$\tilde{u}(z) = -2x_{02} + (2x_{02} - 2x_{01})z + Q(z).$$

For example, for  $N = 1$  the finite control is unique and has the form

$$u(0) = -2x_{02}, \quad u(1) = 2x_{02} - 2x_{01}.$$

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*Note: Figure translations are in progress. See original paper for figures.*

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