ON THE SPECTRUM OF SINGULAR INTEGRALS ON DOMAINS WITH BOUNDARY

MATHEMATICS

1969

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Abstract

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UDC 517.944

MATHEMATICS

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ON THE SPECTRUM OF SINGULAR INTEGRALS ON DOMAINS WITH BOUNDARY

(Presented by Academician S. L. Sobolev on 22 X 1968)

Let Ω be a domain of m-dimensional Euclidean space, bounded by a finite number of simple closed surfaces of Lyapunov type, not intersecting one another. Introduce the operator of extension by zero outside Ω

$$P_\Omega : L_2(\Omega) \ni f(x) \to (P_\Omega f)(x) = \begin{cases} f(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \notin \Omega \end{cases} \in L_2(E_m)$$

and the operator of restriction to Ω

$$R_\Omega : L_2(E_m) \ni f(x) \to (R_\Omega f)(x) = \{f(x) \text{ for } x \in \Omega\} \in L_2(\Omega).$$

Let $\mathcal{A}$ be a singular integral operator with symbol $\sigma(x, \xi)$. The function $\sigma(x, \xi)$ is defined for all x and $\xi \in E_m$, $\xi \neq 0$, is positively homogeneous in ξ of degree zero, and on the unit sphere $\Sigma = \{\xi : |\xi| = 1\}$ belongs to the space $H_2(\Sigma)$. Suppose that, with respect to x, the function $\sigma(x, \xi)$ is sufficiently smooth. As is known, the operator $\mathcal{A}$ maps $L_2(E_m)$ into itself.

The present work is devoted to the study of the essential spectrum of the operator

$$A = R_\Omega \mathcal{A} P_\Omega.$$

Obviously, A maps $L_2(\Omega)$ into itself. A point λ is called a point of the essential spectrum of the operator A if the operator $A - \lambda I$ is not Noetherian.

We shall use the necessary and sufficient condition for the Noetherian property of singular integral operators contained in ([1]).

Let $m \geq 2$, and let x_0 be a point of the boundary of Ω. Draw at x_0 the unit normals: the inward $n^i_{x_0}$ and the outward $n^e_{x_0}$. Move $n^i_{x_0}$ and $n^e_{x_0}$ to the origin of coordinates. Their endpoints mark on the unit sphere Σ two points. Connect these points by all possible semicircles l_{x_0}. Introduce the quantity

$$d_{x_0}^{l_{x_0}}(\lambda) = \{\arg[\sigma(x,\xi) - \lambda]\}_{l_{x_0}}.$$

The braces denote the change of the quantity in the braces when ξ varies along the semicircle l_{x_0}.

In the case $m > 2$ all paths l_{x_0} are homotopic, and $d_{x_0}^{l_{x_0}}(\lambda)$ does not depend on l_{x_0}. We denote the common value $d_{x_0}^{l_{x_0}}(\lambda)$ by $d_{x_0}(\lambda)$.

In the case $m = 2$ there are two classes of nonhomotopic paths, and we obtain two numbers $d_{x_0}^{+}(\lambda)$ and $d_{x_0}^{-}(\lambda)$. In the case $m = 1$ we denote $d_{x_0}(\lambda) = \arg[\sigma(x_0, 1) - \lambda] - \arg[\sigma(x_0, -1) - \lambda]$.

Using the results of ([1]), we obtain the following proposition:

Theorem. *The essential spectrum of the operator A consists of the values of the function $\sigma(x,\xi)$ for $x \in \Omega$, $\xi \in \Sigma$, and of those points λ for which $|d_{x_0}(\lambda)| \geq \pi$ ($|d_{x_0}^{\pm}(\lambda)| \geq \pi$) for some x_0 on the boundary of Ω.*

Example 1. Consider the operator

$$T : L_2(0,1) \ni u(x) \to m(x)u(x) + \int_0^1 \frac{K(x,t)}{x-t}\, u(t)\, dt,$$

where $m(x)$ and $K(x,t)$ have continuous first derivatives. This operator was studied in ([2]). Write T in the form

$$T = A + C,$$

where

$$A : L_2(0,1) \ni u(x) \to m(x)u(x) + k(x) \int_0^1 \frac{u(t)}{x-t}\, dt, \qquad k(x) = K(x,x);$$

$$C : L_2(0,1) \ni u(x) \to \int_0^1 \frac{K(x,t) - K(x,x)}{x-t}\, u(t)\, dt.$$

It is easy to see that C is a completely continuous operator. Consequently, the operators T and A have the same essential spectrum. Let

$$\mathcal{A} : L_2(-\infty,\infty) \ni u(x) \to m(x)u(x) + k(x) \int_{-\infty}^{\infty} \frac{u(t)}{x-t}\, dt.$$

Then

$$A = R_\Omega \mathcal{A} P_\Omega,$$

where $\Omega = (0,1)$. For the symbol of the operator $\mathcal{A}$ we obtain

$$\sigma(x,\xi) = m(x) + i\pi k(x)\,\operatorname{sgn}\xi.$$

The set of values of $\sigma(x,\xi)$ for $x \in (0,1)$ and $|\xi| = 1$ consists of two curves

$$A_1B_1: \ m(x) + i\pi k(x), \qquad 0 \le x \le 1,$$

$$A_2B_2: \ m(x) - i\pi k(x), \qquad 0 \le x \le 1.$$

It is easy to see that the points λ for which $\left|d_{x_0}(\lambda)\right| \ge \pi$ fill two rectilinear segments A_1A_2 and B_1B_2, where

$$A_1 = m(0) + i\pi k(0), \qquad A_2 = m(0) - i\pi k(0),$$

$$B_1 = m(1) + i\pi k(1), \qquad B_2 = m(1) - i\pi k(1).$$

Thus, the essential spectrum of the operator T consists of the points of the curvilinear quadrilateral $A_1B_1A_2B_2$. This agrees with the result ([2]), obtained with the aid of the theory of normed rings.

Example 2. Let $m \ge 2$, and let Ω be a bounded domain in m-dimensional Euclidean space with sufficiently smooth boundary $\partial\Omega$. Denote by $G(x;y)$, $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$, the Green' s function of the problem

$$\Delta u = f, \qquad u|_{\partial\Omega} = 0, \qquad \Delta = \partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_m^2.$$

Consider the operator

$$T : L_2(\Omega) \ni u(x) \to \int_\Omega \frac{\partial^2 G(x;y)}{\partial y_1^2}\, u(y)\, dy.$$

Differentiation under the integral is performed in the space of generalized functions. This operator was studied in ([3–5]).

It is not hard to see that T can be represented in the form

$$T = R_\Omega \mathcal{A} P_\Omega + C,$$

where C is a completely continuous operator, and

$$\mathcal{A} : L_2(E_m) \ni u(x) \to \begin{cases} \int_{E_m} \frac{\partial^2}{\partial y_1^2}(1 - r^{m-2})u(y)\,dy, & \text{if } m > 2, \\ \int_{E_m} \frac{\partial^2}{\partial y_1^2}(\ln r)u(y)\,dy, & \text{if } m = 2. \end{cases}$$

The operator $\mathcal{A}$ has the symbol

$$\sigma(x, \xi) = \xi_1^2 |\xi|^{-2}.$$

Applying the theorem, we obtain that the essential spectrum of the operator T fills the interval $[0, 1]$.

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Received
2 IX 1969

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Note: Figure translations are in progress. See original paper for figures.